

Shell Model for Time-correlated Random Advection of Passive Scalars

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We study a minimal shell model for the advection of a passive scalar by a Gaussian time correlated velocity field. The anomalous scaling properties of the white noise limit are studied analytically. The effect of the time correlations are investigated using perturbation theory around the white noise limit and non-perturbatively by numerical integration. The time correlation of the velocity field is seen to enhance the intermittency of the passive scalar.

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I. INTRODUCTION

The advection of a scalar observable $\theta(x, t)$ by a velocity field \mathbf{v} is described in classical hydrodynamics by the linear PDE

$$\partial_t \theta + \mathbf{v} \cdot \nabla \theta = \kappa \nabla^2 \theta + \mathbf{f}. \quad (1.1)$$

If \mathbf{v} is assumed to be solution of the Navier-Stokes equations in a turbulent régime and the Péclet number Pe , which measures the ratio between the strength of the advective effects and the molecular diffusion κ in (1.1), is large

$$Pe \equiv \frac{Lv}{\kappa} \gg 1$$

(L and v are the characteristic length and advection velocity in the problem) and if a steady state is reached, an inertial range sets in where both the effects of the forcing \mathbf{f} limited to the large scales and those of the molecular diffusion acting mainly on the small scales can be neglected. In the inertial range no typical scale is supposed to characterise the flow. As a consequence, the structure functions of the scalar field

$$S_p(\mathbf{r}) = \langle [\theta(\mathbf{x} + \mathbf{r}) - \theta(\mathbf{x})]^p \rangle \quad (1.2)$$

display a power law behaviour in the inertial range with anomalous scaling exponents $H(p)$ [1]. The word anomalous means that the exponents $H(p)$ deviates from the linear behaviour predicted by a direct scaling analysis of (1.1).

It was first realised by Kraichnan [2] that anomalous scaling can be observed in the mathematically more tractable case of the advection by a random homogeneous and isotropic Gaussian velocity field, which is delta correlated (white noise) in time and has zero average and covariance in d dimensions given by:

$$\langle \mathbf{v}_i(\mathbf{x}, t) \mathbf{v}_j(\mathbf{y}, s) \rangle = \delta(t - s) \left[D_{i,j}(0) - D_0 |\mathbf{x} - \mathbf{y}|^{\xi_{wn}} (d - 1 + \xi_{wn}) \delta_{i,j} + \xi_{wn} \frac{(x - y)_i (x - y)_j}{|\mathbf{x} - \mathbf{y}|^2} \right].$$

The power law behaviour of the covariance mimics an infinite inertial range for the velocity field. The scaling exponent ξ_{wn} is a free parameter characterising the degree of turbulence of the advecting field. The physically meaningful values range from zero to two. In the first limit the effect of the random advection is just to define an effective diffusion constant [3]. In the latter case the velocity increments are smooth as it is expected for a laminar flow. The choice ξ_{wn} equal to four thirds represents the scaling of the velocity field conjectured by Kolmogorov for the solution of the Navier-Stokes equation in turbulent régime.

The hypothesis of delta correlation in time is of great mathematical advantage for it allows to write the equation of motions of the scalar correlations in a linear closed form. The evolution of each correlation in the inertial range is specified by a linear differential operator, the inertial operator, plus matching conditions at the boundary of the inertial range. The occurrence of anomalous scaling has been related to the existence of zero modes of the inertial operators dominating the scaling properties of higher order correlations ([4,5,3,6] and for a recent review and more

complete bibliography [7]). The behaviour of the anomaly has also been numerically measured for the fourth order structure function versus the turbulence parameter ξ_{wn} [8]. However, to implement accurate numerical experiments still remains a difficult task. Therefore it turns out to be useful to use shell model as laboratories to test ideas and results related to the full PDE model (see [10] for a general introduction to the shell model concept). In [16,17] two different shell models advected by a delta correlated velocity field mimicking the Kraichnan model were constructed. Anomalous scaling was observed numerically and in the simpler case [17] it was proven analytically that the anomaly of the fourth order structure function is related to the anomalous scaling of the dominant zero mode of the inertial operator.

The passive scalar advection by a white noise velocity field is a useful mathematical model, but it still very far from a physical realistic velocity field which displays both time correlations and non Gaussian fluctuations. A first small step in this direction is made by investigating how the introduction of a time correlation in a Gaussian velocity field affects the statistical properties of the scalar field.

In the present paper we introduce a time correlated velocity field in a shell model. This is done by substituting the white noise with the Ornstein-Uhlenbeck process which provides exponentially decaying time correlations (Section II). We investigate the model both analytically and numerically. By means of stochastic variational calculus, which we shortly review in Appendices A and B, we show how to rewrite the equations of motion for the scalar correlations in integral non closed form. Such an operation allows the evaluation of the correction to the white noise inertial operator stemming from the time correlated velocity field. This procedure has the further advantage that it creates a non-ambiguous relation between the coupling terms for the scaling exponent ξ_{wn} of white noise advection to the scaling exponent ξ of the equal time correlation of the time-correlated velocity field (Section III).

The inertial operators can be expanded around the white noise limit in powers of an a -dimensional parameter which is interpreted as proportional to the ratio ϵ between the time correlation and the turn-over time of the advecting field. We focus on the features of the steady state. There we assume that the averages over the Ornstein-Uhlenbeck process of all the observables are time translational invariant. As a consequence the inertial operators become linear up to any finite order in ϵ .

In the white noise case, when ϵ is equal zero, we generalise the procedure first introduced in [17] and we show that the scaling of the zero modes of the inertial operator of any order is captured by focusing on nearest-shell interactions. The equations are closed with a scaling Ansatz (section IV) by postulating that the scalar field is “close” to a multiplicative process. Furthermore we perturb the closure scheme in order to extract the first order corrections in ϵ to the anomalous exponents for different values of ξ ranging from zero to two. The prediction of perturbation theory is an ϵ dependence (non-universality) of the exponents except for the second order $H(2)$ (Section V). The overall result is analogous to the one obtained in [14] where a Gaussian time correlated velocity field is considered for the advection of the scalar field in (1.1): the introduction of time correlation is seen to enhance intermittency. The anomalies vanish smoothly in the laminar limit ξ equal two.

To examine the validity of the results from the analytical calculations and explore the regime with long time-correlations ($\epsilon \gg 0$) we turned to numerical experiments. The occurrence of corrections to the anomalies predicted by the perturbation theory for small values of ϵ was confirmed. However, strong non-perturbative effects sets in and are dominating when the expansion parameter becomes of the order of unity.

II. THE MODEL

The model is defined by the equations ($m = 1, 2, \dots, N$)

$$\left[\frac{d}{dt} + \kappa k_m^2\right]\theta_m(t) - \delta_{1m}f(t) = i[k_{m+1}\theta_{m+1}^*(t)u_m^*(t) - k_m\theta_{m-1}^*(t)u_{m-1}^*(t)] \quad (2.1)$$

$$u_m(t) = \frac{v_m}{\epsilon\sqrt{\tau_m}} \int_0^t ds e^{-\frac{t-s}{\epsilon\tau_m}} \eta_m(s) \quad (2.2)$$

$$f(t) = \frac{\tilde{f}}{\epsilon\sqrt{\tau}} \int_0^t ds e^{-\frac{t-s}{\epsilon\tau}} \eta(s) \quad (2.3)$$

where the star denotes complex conjugation, the $\eta_m(t)$'s and $\eta(t)$ are independent white noises with zero mean value and correlation:

$$\langle \eta_m(t)\eta_n^*(s) \rangle = 2\delta_{mn}\delta(t-s) \quad \text{and} \quad \langle \eta(t)\eta^*(s) \rangle = 2\delta(t-s) \quad (2.4)$$

The boundary conditions are $\theta_0 = \theta_{N+1} = 0$. The model might be regarded as a severe truncation of the equation of the passive scalar (1.1) in Fourier space. The field component θ_m is the representative of all the Fourier modes in the

shell with a wavenumber ranging between $k_m = k_0 \lambda^m$ and $k_{m+1} = k_0 \lambda^{m+1}$. The parameter λ is the ratio between two adjacent scales and it is usually taken equal to two in order to identify each shell with an octave of wave numbers. The energy transfer in a turbulent flow is conjectured to occur mainly through the interactions of eddies of the same size. As a consequence the interactions in Fourier space are assumed to be local. The “localness” conjecture [1] is the motivation for the restriction to nearest neighbours of the couplings among the shells.

In the absence of external forcing and dissipation the total “energy” of the passive field is conserved:

$$\frac{d}{dt} E = \frac{d}{dt} \sum_{m=1}^N |\theta_m|^2 = 0 \quad \text{for} \quad f(t) = \kappa = 0. \quad (2.5)$$

Far from the infra-red and the ultra-violet boundaries (i.e. for $1 \ll m \ll N$) the conservation of energy is expected to hold approximately giving rise to an inertial range. Equations (2.2) and (2.3) describe the random evolution according to Ornstein-Uhlenbeck (O-U) processes respectively of the advecting and external force field. The O-U process has differentiable realisations thus making the random differential equations with multiplicative noise which specify the dynamics of the scalar θ independent of the discretisation prescription.

The velocity correlations are for $t \geq s$

$$\langle u_m(t) u_m^*(s) \rangle = \frac{|v_m|^2}{\epsilon} (e^{-\frac{t-s}{\epsilon \tau_m}} - e^{-\frac{t+s}{\epsilon \tau_m}}). \quad (2.6)$$

In the limit of large t only the stationary part survive. The a-dimensional parameter ϵ appearing in the definition of the O-U processes (2.2) and (2.3) defines the strength of the time correlation in units of the typical times τ_m . In the white noise limit one has

$$\lim_{\epsilon \downarrow 0} \langle u_m(t) u_m^*(s) \rangle = 2|v_m|^2 \delta\left(\frac{t-s}{\tau_m}\right). \quad (2.7)$$

The factor two is reintroduced here with the proviso that the delta must be understood according to the midpoint prescription when its argument coincides with the upper boundary of integration as it occurs in practical computations in perturbation theory around the white noise limit.

For any finite ϵ ordinary differential calculus holds true: the consistency conditions yields a Stratonovich discretisation prescription when ϵ is set to zero and the recovery of the white noise advection model of [17].

The information about the scaling of the correlations of the velocity field at equal times is stored in the constants v_m . We assume the power law behaviour

$$|v_m| \propto k_m^{-\frac{\xi}{2}}. \quad (2.8)$$

Kolmogorov scaling is specified by $\xi = 2/3$ while $\xi = 2$ corresponds to a laminar regime. The τ_m 's in (2.2) describe the typical correlation times for the random velocity field. A simple physical interpretation is to identify them with the turn-over times, i.e., with the typical time rates of variation through non linearity of the advection field on each shell [9]:

$$\tau_m \sim \frac{1}{k_m |v_m|} \propto k_m^{-1+\frac{\xi}{2}}. \quad (2.9)$$

The scaling of the correlation times is then fully specified in terms of the parameter ξ . It is worth to note that for any ξ less than two the τ_m 's are always decreasing functions of the wave number.

The evolution of the scalar θ is determined in the inertial range by its complex conjugate. It is useful to introduce an unified notation for the $2N$ degrees of freedom. Calling $\Theta = \theta \oplus \theta^*$ and $U = u \oplus u^*$ one has for the N shells:

$$\frac{d}{dt} \Theta_\alpha = \sum_{\beta=1}^{2N} [A_{\alpha,\beta} + \sum_{\gamma=1}^{2N} B_{\alpha,\beta}^\gamma U_\gamma] \Theta_\beta + f \delta_{\alpha,1} + f^* \delta_{\alpha,N+1} \quad (2.10)$$

with

$$\begin{aligned} A_{m,\beta} &= -\kappa k_m^2 \delta_{m,\beta} \\ A_{N+m,\beta} &= -\kappa k_m^2 \delta_{m,n} \\ B_{\alpha,\beta}^m &= -ik_{m+1} [\delta_{\beta,m+1} \delta_{\alpha,N+m} - \delta_{\beta,m} \delta_{\alpha,N+m+1}] \\ B_{\alpha,\beta}^{N+m} &= ik_{m+1} [\delta_{\beta,N+m+1} \delta_{\alpha,m} - \delta_{\beta,N+m} \delta_{\alpha,m}] \end{aligned} \quad (2.11)$$

where Latin and Greek indices range respectively from 1 to N and from 1 to $2N$. The set of matrices with constant entries B^γ do not commute within each other and with the A matrix. The known sufficient condition (see for example [15]) to have a solution of (2.10) in an analytic exponential form is therefore not satisfied. From the geometrical point of view non-commutativity means that the dynamics is confined on a manifold which turns into an hyper-sphere in \mathcal{C}^N in the inertial limit (2.5).

The complex equations (2.10) are invariant under phase transformations. Given two diagonal hermitian $2N \times 2N$ matrices with time independent random entries

$$T \equiv \text{diag}(e^{i\phi_1}, \dots, e^{i\phi_N}, e^{-i\phi_1}, \dots, e^{-i\phi_N}) \quad (2.12)$$

$$S \equiv \text{diag}(e^{-i(\phi_1+\phi_2)}, \dots, e^{-i(\phi_{N-1}+\phi_N)}, 0, e^{i(\phi_1+\phi_2)}, \dots, e^{i(\phi_{N-1}+\phi_N)}, 0) \quad (2.13)$$

if Θ is a realisation of the solution of the equations of motion then

$$T\Theta(U) = \Theta(SU) \quad (2.14)$$

is still a solution. The phase symmetry is the remnant of the translational invariance of the original hydrodynamical equations in real space [10]. From the phase symmetry (2.14) it follows that at stationarity the only analytic non zero moments of the correlation are of the form

$$C_{m_1, \dots, m_\omega}^{(2\omega)} = \langle \Pi_{i=1}^\omega \Theta_{m_i} \Theta_{N+m_i} \rangle \equiv \langle \Pi_{i=1}^\omega |\theta_{m_i}|^2 \rangle. \quad (2.15)$$

In the inertial range such quantities display a power law behaviour. The diagonal sector of the moments whose scaling properties are specified by the exponents $H(2\omega)$

$$C_{m, \dots, m}^{(2\omega)} \propto k_m^{-H(2\omega)} \quad (2.16)$$

is in the shell model context the analogue of the structure functions (1.2) of the original PDE model (1.1). The exponents $H(2\omega)$'s are said to be normal if they can be derived from dimensional analysis. Under the assumption that a steady state is reached one matches the scaling of the inertial terms in (2.1) with a power law decay of the solution

$$k_{m+1} k_m^{-\frac{\xi}{2}} \theta_{m+1} - k_m k_{m-1}^{-\frac{\xi}{2}} \theta_{m-1} \sim 0. \quad (2.17)$$

The resulting prediction is a linear behaviour of the exponents versus the order ω of the diagonal correlation:

$$H(2\omega) = \omega \left(1 - \frac{\xi}{2}\right). \quad (2.18)$$

The scaling argument (2.17) neglects completely the random fluctuations of the passive scalar field. Normal scaling holds if the statistics of the θ -field is Gaussian. Deviations from normal scaling are then related to the occurrence of intermittency corrections to the Gaussian statistics. A systematic account of the fluctuations is provided by the study of the equations of motion satisfied by the moments of the scalar field.

III. EQUATIONS OF MOTION OF THE FIELD MOMENTS

In the white noise limit, ϵ equal zero, the Furutsu-Donsker-Novikov formula [1] and the delta correlation in time of the velocity ensure that the moments $C^{(2\omega)}$ are specified by the solutions of closed linear systems [16,17]. In the presence of finite time correlations stochastic calculus of variations [18,19] allows to write non closed integro-differential equations for the correlations. A typical functional integration by parts relation is:

$$\langle F(\Theta(t)) U_{N+m}(t) \rangle = \int_0^t ds \langle U_{N+m}(t) U_m(s) \rangle \left\langle \frac{dF(\Theta(t))}{d\Theta_\alpha(t)} R_{\alpha,\beta}(t,s) B_{\beta,\gamma}^m \Theta_\gamma(s) \right\rangle \quad (3.1)$$

where Einstein convention holds for repeated *Greek* indices. The matrix R is the fundamental solution of the homogeneous system associated with (2.10). A heuristic proof of the stochastic integration by parts formula and of (3.1) is provided in Appendices A and B.

Let us start with the second moment of the scalar field

$$C_m^{(2)}(t) \doteq \langle \Theta_m(t) \Theta_{N+m}(t) \rangle \equiv \langle \theta_m(t) \theta_m^*(t) \rangle. \quad (3.2)$$

From the equations of motion (2.10) one has

$$\begin{aligned} & \left[\frac{d}{dt} + 2\kappa k_m^2 \right] C_m^{(2)}(t) - 2\Re \{ \langle \Theta_{N+m}(t) f(t) \rangle \} \delta_{m,1} = \\ & + 2\Re \{ i k_{m+1} \langle U_{N+m}(t) \Theta_{N+m+1}(t) \Theta_{N+m}(t) \rangle \} + \\ & - 2 k_m \Re \{ i \langle U_{N+m-1}(t) \Theta_{N+m-1}(t) \Theta_{N+m-1}(t) \rangle \}. \end{aligned} \quad (3.3)$$

The integration by parts formula (3.1) gives

$$\begin{aligned} & \left[\frac{d}{dt} + 2\kappa k_m^2 \right] C_m^{(2)}(t) - 2\delta_{m,1} \Re \int_0^t ds \langle f(t) f(s)^* \rangle \langle R_{N+m,N+1}(t, s) \rangle = \\ & 2 k_{m+1}^2 \tau_m \int_0^t ds \frac{\langle U_m(t) U_{N+m}(s) \rangle}{\tau_m} \Re \mathcal{F}_m^{(2)}(t, s) + \\ & - 2 k_m^2 \tau_{m-1} \int_0^t ds \frac{\langle U_{m-1}(t) U_{N+m-1}(s) \rangle}{\tau_{m-1}} \Re \mathcal{F}_{m-1}^{(2)}(t, s) \end{aligned} \quad (3.4)$$

where $m = 1, \dots, N$, \Re is the real part and:

$$\begin{aligned} & \mathcal{F}_m^{(2)}(t, s) \doteq G_{N+m+1, N+m; N+m, m+1}^{(2)}(t, s) - G_{N+m+1, N+m+1; N+m, m}^{(2)}(t, s) \\ & + G_{N+m, N+m; N+m+1, m+1}^{(2)}(t, s) - G_{N+m, N+m+1; N+m+1, m}^{(2)}(t, s) \end{aligned} \quad (3.5)$$

$$G_{N+m, N+n; N+p, q}^{(2)}(t, s) \doteq \sum_{\alpha=1}^{2N} \left\langle \Theta_{N+p}(t) R_{N+m+1, \alpha}(t, 0) R_{\alpha, N+n}^{-1}(s, 0) \Theta_q(s) \right\rangle \quad (3.6)$$

$$d_m \doteq |v_m|^2 \tau_m \propto k_m^{-(1+\frac{\xi}{2})} \quad (3.7)$$

When a steady state is reached the LHS of (3.4) can be neglected through the whole inertial range. The RHS specifies the inertial operator of the theory. A further simplification is attained in the limit of very large shell number. For any ξ less than two one has

$$\lim_{m \uparrow \infty} \frac{\langle U_m(t) U_{N+m}(s) \rangle}{\tau_m} \equiv \lim_{m \uparrow \infty} \frac{\langle u_m(t) u_m^*(s) \rangle}{\tau_m} = |v_m|^2 \delta(t-s) \quad (3.8)$$

independently of ϵ . At equal times the resolvent matrix R reduces to the identity. From (2.8) and (2.9) it stems that

$$k_{m+1}^2 d_m \tau_m = \lambda^2. \quad (3.9)$$

Hence for m going to infinity the inertial operator is linearised in the form

$$RHS = 2 \frac{\lambda^2}{\tau_m} \left(C_{m+1}^{(2)} - C_m^{(2)} \right) - 2 \frac{\lambda^2}{\tau_{m-1}} \left(C_m^{(2)} - C_{m-1}^{(2)} \right). \quad (3.10)$$

The slowest decay scaling solution compatible with a zero LHS is

$$C_m^{(2)} \propto \tau_m = k_m^{-H(2)}, \quad (3.11)$$

In other words we have proven that the scaling of the second moment is normal as it coincides with the dimensional prediction (2.18). Moreover since the result does not depend on ϵ it is universal versus the time correlation. It is worth stressing that the derivation of (3.11) requires that each of the terms appearing in (3.10) has separately a finite non zero limit for m going to infinity. The condition turns out to be not self-consistent when the same reasoning is applied to moments higher than the second.

An important consequence of normal scaling of the $C_m^{(2)}$'s is the Obukhov-Corrsin [11,12] law for the decay of the power spectrum $\Gamma(k)$ of the passive scalar if the Kolmogorov scaling is assumed for the advecting field:

$$\Gamma(k) = \frac{d}{dk} \sum_{k_n \leq k} \langle (\theta_n \theta_n^*)^2 \rangle \propto k^{-(H(2)+1)}|_{\xi=2/3} = k^{-\frac{5}{3}}. \quad (3.12)$$

A second interesting limit is when ϵ tends to zero. Neglecting all non stationary contributions to the velocity correlations the RHS of (3.4) becomes

$$\begin{aligned}
RHS = & 2k_{m+1}^2 d_m \left(C_{m+1}^{(2)} - C_m^{(2)} \right) - 2k_m^2 d_{m-1} \left(C_m^{(2)} - C_{m-1}^{(2)} \right) + \\
& -2k_{m+1}^2 d_m \int_0^t ds e^{-\frac{t-s}{\epsilon \tau_m}} \frac{d}{ds} \Re \mathcal{F}_m^{(2)}(t, s) + \\
& + k_m^2 d_{m-1} \int_0^t ds e^{-\frac{t-s}{\epsilon \tau_{m-1}}} \frac{d}{ds} \Re \mathcal{F}_{m-1}^{(2)}(t, s).
\end{aligned} \tag{3.13}$$

If ϵ is set exactly to zero the integral terms disappear and the white noise equations of [17] are recovered. The information about the scaling of the velocity field is absorbed in the d_m 's. In a pure white noise theory it is convenient to redefine the turbulence parameter as

$$\xi_{wn} = 1 + \frac{\xi}{2}. \tag{3.14}$$

A Kolmogorov scaling of the velocity field corresponds to ξ_{wn} equal to four thirds which is also the value giving the Obukhov-Corrsin scaling in (3.12). The two definitions of the degree of turbulence coincide for ξ equal to two (Batchelor limit). It is natural to identify ξ_{wn} with the turbulence parameter of the Kraichnan model. The correspondence fixes the physical range of ξ between $[-2, 2]$.

In the general case of the 2ω -th even moment of the scalar $C^{(2\omega)}$ one has:

$$\begin{aligned}
RHS = & \sum_{q_1, \dots, q_\omega} I_{m_1, \dots, m_\omega, q_1, \dots, q_\omega}^{(2\omega; 0)} C_{q_1, \dots, q_\omega}^{(2\omega)} + \\
& - \sum_{i=1}^{\omega} 2k_{m_i+1}^2 d_{m_i} \int_0^t e^{-\frac{t-s}{\epsilon \tau_{m_i}}} \frac{d}{ds} \Re \mathcal{F}_{m_1, \dots, m_i, \dots, m_\omega}^{(2\omega)}(t, s) + \\
& + \sum_{i=1}^{\omega} 2k_{m_i}^2 d_{m_i-1} \int_0^t ds e^{-\frac{t-s}{\epsilon \tau_{m_i-1}}} \frac{d}{ds} \Re \mathcal{F}_{m_1, \dots, m_i-1, \dots, m_\omega}^{(2\omega)}(t, s).
\end{aligned} \tag{3.15}$$

The multidimensional matrix $I^{(2\omega; 0)}$ is the linear inertial operator of the white noise theory. The integrand functions $\mathcal{F}_{m_1, \dots, m_i-1, \dots, m_\omega}^{(2\omega)}(t, s)$ are given by the straightforward generalisation of (3.5). The LHS, as above, is set to zero as far as the steady state features of the inertial range are concerned. Repeated integrations by parts in the large t limit generate a Laplace asymptotic expansion [13] of integral terms in the RHS, the coefficient of which are the derivatives with respect to s of the functions $\mathcal{F}^{(2\omega)}$ evaluated at equal times. When the steady state sets in we assume the latter quantities to be invariant under time translations for large t . Under such an assumption it will be proven in section V that the equal time derivatives are specified at equilibrium by linear combinations of the $C^{(2\omega)}$'s. The effect of a small time correlation is therefore to generate new couplings of order ϵ in the inertial operators. The observables we focus on are the scaling exponents. As discussed in the introduction, anomalies occur in the presence of non trivial scaling zero modes of the white noise inertial operators. It makes sense to relate the ϵ dependence of the anomalous exponents to a perturbation of the scaling zero modes derived for ϵ equal zero. A straightforward approach to the problem calls for the solution of N^ω linear equations. A further source of difficulty is that the exact determination of the zero eigenvectors of the inertial operators of any given order requires the matching of infra-red and ultra-violet boundary conditions. In the absence of an exact diagonalisation any analytical approach must rely on closure Ansätze to solve first the white noise problem and then to yield the corrections to the zero modes by linear perturbation theory.

IV. THE WHITE NOISE CLOSURE

In the present section we present a closure strategy to compute the $H(2\omega)$'s in the case of white noise advection. As shown in the previous section the second diagonal moment is normal and universal versus the time correlation. The first nontrivial zero mode problem is provided by the fourth order inertial operator $I^{(4; 0)}$. In [17] it has been shown that the anomalous exponent ρ_4

$$H(4) = 2H(2) - \rho_4 \tag{4.1}$$

can be extracted up to a very good accuracy from the solution of only two non-linear algebraic equations. The stationary equations for $C^{(4)}$ in the inertial range far from the infra-red and ultra-violet boundaries are given by

$$\begin{aligned}
0 &= \frac{I_{m,n;p,q}^{(4;0)} C_{p,q}^{(4)}}{2\lambda^2} \equiv \\
&\equiv -\left(\frac{1}{\tau_m} + \frac{1}{\tau_{m-1}} + \frac{1}{\tau_n} + \frac{1}{\tau_{n-1}}\right) C_{m,n}^{(4)} + \frac{1}{\tau_m} C_{m+1,n}^{(4)} + \frac{1}{\tau_n} C_{m,n+1}^{(4)} + \\
&+ \frac{1}{\tau_{m-1}} C_{m-1,n}^{(4)} + \frac{1}{\tau_{n-1}} C_{m,n-1}^{(4)} + 2\delta_{m,n} \left(\frac{C_{m,m+1}^{(4)}}{\tau_m} + \frac{C_{m,m-1}^{(4)}}{\tau_{m-1}}\right) + \\
&- 2\delta_{m+1,n} \frac{C_{m,m+1}^{(4)}}{\tau_m} - 2\delta_{n,m-1} \frac{C_{m,m-1}^{(4)}}{\tau_{m-1}}.
\end{aligned} \tag{4.2}$$

One recognises two kinds of couplings in $I_{m,n;p,q}^{(4;0)}$.

- “Global” or “unconstrained” interactions: the indices p and q range respectively from $m-1$ to $m+1$ and from $n-1$ to $n+1$. The coupling are independent upon the relative values of m and n : in this sense they are referred as global.
- “Purely local” interactions: they occur only for $|m-n| \leq 1$ and correspond to the terms proportional to the Kroenecker’s δ in (4.2).

Anomalous scaling in the inertial range is strictly related to the presence of such purely local interactions. Were these latter neglected the fourth order moment would have a normal scaling solution

$$C_{m,n}^{(4)} \propto \frac{\tau_n}{\tau_m} \tau_m^2. \tag{4.3}$$

The idea is to capture the anomalous scaling by looking at the “renormalisation” of global couplings by pure short range ones. Disregarding the boundaries, the system is invariant under a simultaneous shift of the indices. Hence assuming a perfect index-shift invariance there are, for the m -th shell, only two independent equations where δ -like terms occur:

$$\begin{aligned}
0 &= \sum_{p,q} I_{m,m;p,q}^{(4;0)} C_{p,q}^{(4)} \\
0 &= \sum_{p,q} I_{m,m-1;p,q}^{(4;0)} C_{p,q}^{(4)}.
\end{aligned} \tag{4.4}$$

The third equation involving a purely local interaction of the m -th shell with its nearest neighbours

$$0 = \sum_{p,q} I_{m+1,m;p,q}^{(4;0)} C_{p,q}^{(4)}.$$

is generated from the second of the (4.4)’s by a simple index shift. Therefore it is not regarded as independent. The pair (4.4) contains all the relevant information to extract the scaling of the fourth moment. It forms a closed system of equations independently on the shell number m as one imposes scaling relations to hold within the set of “independent” moments of fourth order:

$$C_{m+n,m+n}^{(4)} = z^{-n} C_{m,m}^{(4)} \tag{4.5}$$

$$C_{m+n,m}^{(4)} = x k_{n-1}^{-H(2)} C_{m,m}^{(4)} \tag{4.6}$$

where the integer n is taken larger than zero. As in the analysis of the interactions the concept of independence stems from the assumption of index shift invariance: the moments of the form $C_{m-n,m}^{(4)}$ are immediately reconstructed once (4.5) and (4.6) are given:

$$C_{m-n,m}^{(4)} = x k_{n-1}^{-H(2)} z^n C_{m,m}^{(4)}$$

Let us analyse the closure Ansatz in more detail. The first equation (4.5) is a global scaling assumption of the “diagonal” sector of the fourth moment. Its justification lies in the very definition of an inertial range. The second scaling assumption relates the diagonal sector to the non diagonal one via a marginal scaling. It is the analogous in the present context of an operator product expansion (OPE) in statistical field theory [20]. There renormalisation group (RG) techniques are able to describe the scaling behaviour of correlations of fields sampled at large real space

distances one from the other. If an observable requires the evaluation of a correlation including the products of one field in two points at short distance i.e. $\langle \phi(x-dx)\phi(x+dx) \dots \rangle$ the RG procedure cannot be directly applied. The problem is overcome by an OPE or short distance expansion. The prescription is to rewrite the product via a Taylor expansion in terms of local composite operators sampled just in one point. Such a point is now well separated from all the other appearing in the correlation function. The original correlation is substituted by a set correlations such that RG applies provided an extra renormalisation, renormalisation of composite operators (RCO), is introduced. The latter is understood by observing that in our example the first term in the Taylor expansion gives

$$\phi(x+dx)\phi(x-dx) \sim \phi(x)^2$$

The mathematical meaning of a field is that one of an operator-valued distribution. The product of two distributions at equal points i.e. $\phi(x)^2$ requires a regularisation before the cut-off is removed in order to make itself sense as a distribution. This is the content of the RCO. Finally at leading order the relation between the renormalised quantities reads for the above example

$$\langle [\phi(x+dx)\phi(x-dx)]_R \dots \rangle \sim c(dx) \langle [\phi(x)^2]_R \dots \rangle. \quad (4.7)$$

Roughly speaking the small real space separations are associated with the UV behaviour of the Fourier conjugated variable. In the shell model context the θ_m are representative of the scalar field variation over one octave. The moments $C_{m,m+n}^{(4)}$ correspond to the Fourier transform of the fourth order structure functions of a homogeneous and isotropic real space solution of the passive scalar equation (1.1). The meaning of the non-diagonal closure (4.6) is to assume the long range (many shells) behaviour of the Fourier transform of the OPE coefficient $c(dx)$ to scale inside the inertial range for large n independently on m . The constant x renormalises the value of the first shell where deviation from scaling occur. The analogy with the OPE is then summarised by writing

$$\begin{aligned} \phi(x) &= [\theta(x) - \theta(0)]_R^2 \\ \lim_{|k| \uparrow \infty} \int d^D x e^{ik \cdot dx} c(dx) &\sim x k_{n-1}^{-H(2)} \end{aligned} \quad (4.8)$$

The insertion of the scaling Ansatz in (4.4) leaves a non-linear system in the unknown variables z and x . By applying the definition $k_n = \lambda^n$ one gets

$$\begin{cases} -1 - \lambda^{-H(2)} + 2x(1 + z\lambda^{-H(2)}) = 0 \\ (1+z)\lambda^{-H(2)} + xz(-1 - 3\lambda^{-H(2)}\lambda^{-2H(2)} + z\lambda^{-3H(2)}) = 0 \end{cases} \quad (4.9)$$

which after a straightforward manipulation provides z as the physical root of a second order polynomial

$$z = \frac{1 + 2\lambda^{-H(2)} + 2\lambda^{-2H(2)} + \lambda^{-3H(2)} + \sqrt{1 + 4\lambda^{-H(2)} + 8\lambda^{-2H(2)} - 6\lambda^{-3H(2)} - 4\lambda^{-5H(2)} + \lambda^{-6H(2)}}}{2(2\lambda^{-2H(2)} + \lambda^{-3H(2)} + \lambda^{-4H(2)})}. \quad (4.10)$$

In terms of z the anomaly is

$$\rho_4 = 2H(2) - \frac{\log z}{\log \lambda} \quad (4.11)$$

and it proves to be in fair agreement with the values obtained from the numerical solution of the exact equations (4.2) [17] and from the numerical integration of (2.1) for all the values of the turbulent exponent ξ in the physical range $[0, 2]$ (see also Fig. 2). The sign of ρ_4 is always positive: the effect of the anomaly is to decrease the diagonal scaling exponent.

The procedure presented in detail for the computation of the fourth order exponent is straightforwardly extended to any higher order moment when one recognises that two crucial observation hold in general.

- In the absence of pure short range couplings the normal scaling prediction holds true far from the boundaries for the zero modes of the inertial operators of any order 2ω .
- For any fixed shell m there is a one to one correspondence between the number of independent equations and moments of order 2ω .

In the case of $C^{(2\omega)}$ there are $2^{\omega-1}$ equations: for any fixed reference shell m_1 the interaction with the second index m_2 is affected by a pure short range coupling if the latter is equal or one unit different from m_1 i.e. only two possible choices and so on until the ω -th index is reached. On the other hand $2^{\omega-1}$ is the number of exponents which characterise the scaling of the 2ω -th moment. The Ansatz is that the marginal scaling of the non-diagonal sector is fully specified in terms of the diagonal scaling exponents of order less than 2ω . By means of the ‘‘OPE’s’’ one is able to close the zero mode equations in terms of $2^{\omega-1}$ unknown renormalisation constants and $H(2\omega)$. The analogy with a field theoretical OPE goes to show that the need for an infinite set of constant does not necessarily imply the non-renormalisability of the real space theory mimicked by the shell model [21].

More concretely the diagonal scaling exponent of the sixth moment ($\omega = 3$) of the scalar field

$$C_{m,n,p}^{(6)} = \langle \Theta_{N+m} \Theta_m \Theta_{N+n} \Theta_n \Theta_{N+p} \Theta_p \rangle \equiv \langle |\theta_m|^2 |\theta_n|^2 |\theta_p|^2 \rangle \quad (4.12)$$

according to the above criterion requires four independent equations (see appendix D)

$$\begin{cases} \sum_{p,q,r} I_{m,m,m;p,q,r}^{(6;0)} C_{p,q,r}^{(6)} = 0 \\ \sum_{p,q,r} I_{m,m,m-1;p,q,r}^{(6;0)} C_{p,q,r}^{(6)} = 0 \\ \sum_{p,q,r} I_{m,m-1,m-1;p,q,r}^{(6;0)} C_{p,q,r}^{(6)} = 0 \\ \sum_{p,q,r} I_{m,m+1,m-1;p,q,r}^{(6;0)} C_{p,q,r}^{(6)} = 0 \end{cases} \quad (4.13)$$

The OPE inspired closure yields

$$\begin{aligned} C_{m+n,m+n,m+n}^{(6)} &= z^{-l} C_{m,m,m}^{(6)} \\ C_{m+n,m+n,m}^{(6)} &= x_1 k_{n-1}^{-H(4)} C_{m,m,m}^{(6)} \\ C_{m+n+p,m+n,m}^{(6)} &= x_2 k_{p-1}^{-H(2)} k_{n-1}^{-H(4)} C_{m,m,m}^{(6)} \\ C_{m+n,m,m}^{(6)} &= x_3 k_{n-1}^{-H(2)} C_{m,m,m}^{(6)} \end{aligned} \quad (4.14)$$

Inserting the ‘‘OPE’’ in (4.13) one gets into the algebraic system for the unknown renormalisation constants (x_1, x_2, x_3) and the diagonal scaling factor z .

$$\begin{aligned} -1 + \lambda^\alpha (-1 + 3z x_1) + 3x_3 &= 0 \\ -\lambda^{2\alpha} z x_1 + \lambda^{4\alpha+\rho_4} z^2 x_1 - 2z (x_1 - 2x_2) + \lambda^\alpha (1 + z (-7x_1 + 4x_3)) &= 0 \\ \lambda^\alpha (1 + 4x_1 - 6x_3) + \lambda^{2\alpha} (4z x_2 - 2x_3) - x_3 &= 0 \\ \lambda^{3\alpha+\rho_4} z x_1 + \lambda^\alpha z (x_1 - 3x_2) - z x_2 + \lambda^{5\alpha+\rho_4} z^2 x_2 - \lambda^{3\alpha} z (x_2 - x_3) + \lambda^{2\alpha} (-4z x_2 + x_3) &= 0 \end{aligned} \quad (4.15)$$

After some algebra (4.15) reduces to a single third order polynomial specifying the physical root of z . It is worth to remark that from the functional dependence of the coefficient of (4.15) the exponent $H(6)$ depends upon the anomaly of $H(4)$. Once again the anomaly evaluated from

$$\rho_6 = 3H(2) - \frac{\log z}{\log \lambda} \quad (4.16)$$

is in fair agreement with numerics (see Fig. 2) for different values of ξ .

In appendix E the same steps are performed in the case of the eight moment $C_{m,n,p,q}^{(8)}$. The analytical predictions for the anomalous exponents are summarised in Fig. 1. In all cases the anomalies are decreasing functions of the turbulence parameter ξ vanishing smoothly when the laminar limit (ξ equal two) is approached. The anomaly of the fourth order moment can be compared with the results of numerical experiments for the fourth order structure function of the Kraichnan model [8]. There the adopted turbulence parameter is ξ_{wn} . For values of ξ_{wn} of order one, i.e., from the Kolmogorov scaling up to the Batchelor limit one indeed observes the same monotonically decreasing behaviour with values of the anomaly of the same order of those found in the shell model. For lower values of ξ_{wn} the anomaly in the Kraichnan model display a maximum before decreasing to zero for ξ_{wn} equal to zero, i.e., when ξ tends to minus two. No sign of such behaviour is observed in shell model. The discrepancy might be an artifact of the shell model, which was originally designed to mimic the supposed local in scale character of the nonlinear interactions in a turbulent flow [10], fails to describe a régime where strong non local effects become important.

On a phenomenological level the energy transfer in the inertial range of turbulent field is related to the occurrence of a cascade mechanism as firstly conjectured by Richardson [22]. The conservation of energy in the inertial range

imposes that the forcing occurring on large real space scales is transferred to small scales (i.e. large wave numbers) before being dissipated. A mathematical description of the cascade is provided by multiplicative stochastic processes [23]. Multiplicative modelling has been shown to account for most of the features observed in real and synthetic turbulence [24,10]. In the present case the idea of a multiplicative structure is incorporated in the hypothesis that the scaling of the non diagonal sector of a given moment of order 2ω is reconstructed once the scaling of the lower moments is known. Such an assumption together with the analysis of the couplings in the inertial operator of order 2ω yields with fair accuracy the scaling exponents of the model without the resort to an exact diagonalisation of inertial operator.

V. PERTURBATIVE ANALYSIS

Let us now turn to the time correlated case. The idea is to evaluate the scaling behaviour of the dominant zero modes of the inertial operators (3.15) linearised up to first order in ϵ by perturbing the white noise closure Ansatz.

The first order corrections in ϵ to the inertial operators are obtained by truncating the integration by parts to the terms linear in ϵ

$$\begin{aligned} 2k_{m_i+1}^2 d_{m_i} \int_0^t e^{-\frac{t-s}{\epsilon\tau_{m_i}}} \frac{d}{ds} \Re \mathcal{F}_{m_1, \dots, m_i, \dots, m_\omega}^{(2\omega)}(t, s) = \\ = 2\epsilon \lambda^2 \frac{d}{ds} \Re \mathcal{F}_{m_1, \dots, m_i, \dots, m_\omega}^{(2\omega)}(t, s)|_{s=t} + O(\epsilon^2, \epsilon e^{-\frac{t}{\epsilon\tau_{m_i}}}). \end{aligned} \quad (5.1)$$

The use of (3.9) in the RHS stresses that the effective a-dimensional expansion parameter is $\epsilon\lambda^2$: the range of reliability of first order perturbation theory is compressed to $\epsilon \leq \lambda^{-2}/10$. As mentioned in section III, in the limit t going to infinity, one expects two time quantities to be stationary. In such a case the derivative with respect to the variable s can be interchanged with the derivative with respect to t and one can use the equations of motion to evaluate (5.1). A direct differentiation with respect to s is consistently taken with respect to the system of stochastic differential equations conjugated by time reversal of equations (2.1) to (2.3). The latter operation in general requires the knowledge of the probability density of the forward problem. In the stationary limit the time reversal operation for the O-U process reduces to the inversion of the sign of the drift term as in the deterministic case. After a slightly more lengthy algebra the result is equal to the differentiation with respect to t with opposite sign.

The computations in the general case are very cumbersome (see appendices C, D and E). It is convenient to exemplify the procedure in the simpler case of the second order correlation. There are four contributions to $\Re \mathcal{F}_m^{(2)}$:

$$\begin{aligned} \frac{d}{dt} G_{N+m+1, N+m; N+m, m+1}^{(2)}(t, s)|_{t=s} &= 0 \\ \frac{d}{dt} G_{N+m+1, N+m+1; N+m, m}^{(2)}(t, s)|_{t=s} &= -\kappa k_{m+1}^2 C_m^{(2)}(t) + \langle \dot{\Theta}_{N+m}(t) \Theta_m(t) \rangle \\ \frac{d}{dt} G_{N+m, N+m+1; N+m, m+1}^{(2)}(t, s)|_{t=s} &= 0 \\ \frac{d}{dt} G_{N+m, N+m; N+m+1, m+1}^{(2)}(t, s)|_{t=s} &= -\kappa k_m^2 C_m^{(2)}(t) + \langle \dot{\Theta}_{N+m+1}(t) \Theta_{m+1}(t) \rangle. \end{aligned} \quad (5.2)$$

By definition

$$\begin{aligned} \langle \dot{\Theta}_{N+m}(t) \Theta_m(t) \rangle &= \frac{1}{2} \frac{d}{dt} \langle |\theta_m(t)|^2 \rangle \\ \langle \dot{\Theta}_{N+m+1}(t) \Theta_{m+1}(t) \rangle &= \frac{1}{2} \frac{d}{dt} \langle |\theta_{m+1}(t)|^2 \rangle. \end{aligned}$$

The time derivative of $\Re \mathcal{F}_{m-1}^{(2)}$ is derived by a simple index shift. As the terms non-diagonal in the resolvent R indices are zero, the second moment inertial operator is not affected by first order perturbation theory. The result is not surprising. The second moment has only one free index. Hence at any order of perturbation theory only global coupling can be generated which are forced by the symmetries of the model to be consistent with a normal scaling of the zero mode. Moreover the Θ_m 's components of the scalar evolve only through the coupling with their complex conjugated Θ_{N+m} 's: their variation is a second order effect in ϵ . The only possible non zero corrections are viscous and can be consistently neglected.

Let us turn to draw the general picture when ω is larger than one. Once again the phase symmetries (2.14) and the fact that when $\mathcal{R}\mathcal{F}_{m_1, \dots, m_i, \dots, m_\omega}^{(2\omega)}$ is known all other terms are yielded by index shift or exchange operations, prevent the corrections to the global couplings from affecting the scaling properties: the resulting “global” sectors of inertial operators have a normal scaling zero mode. This is in agreement with the observation made in [14], where the dependence of the scaling exponents on the time correlation for generalised models of passive scalar advection is predicted to appear only through anomalies. The corrections to the purely short range couplings are therefore the relevant ones. They occur in two ways. On one hand new terms of order ϵ show up in the purely self and nearest-neighbours interactions. On the other hand, terms proportional to $\delta_{m_i, m_j \pm 2}$ appear. The latter ones are the most dangerous for they in principle perturb the logic of the white noise closure by introducing new independent equations and henceforth the need for more renormalisation constants in the non diagonal sector of the moments. Nevertheless one can argue a priori in the spirit of the renormalisation group [25], only the nearest neighbours interactions are relevant for scaling. Hence first order corrections can be obtained allowing an ϵ dependence in the renormalisation constant of the white noise closure and determining the first order coefficient of their Taylor expansion. Moreover for ω larger than two such a strategy is already able to take into account the corrections due to the purely second neighbours interactions.

Let us analyse in more detail the case of $C_{m,n}^{(4)}$. The white noise closure is perturbed by introducing an ϵ dependence in the renormalisation constants

$$C_{m+n, m+n}^{(4)} = z(\epsilon)^{-n} C_{m,m}^{(4)} \quad (5.3)$$

$$C_{m+n, m}^{(4)} = x(\epsilon) \lambda_{n-1}^{-H(2)} C_{m,m}^{(4)}. \quad (5.4)$$

The marginal scaling in the non-diagonal sector in (5.4) is assumed to stay universal as it is for $C^{(2)}$ while the ϵ dependence is stored in the pre-factor. The diagonal exponent is then determined up to first order as

$$H(4, \epsilon) = \frac{\log(z)}{\log(\lambda)} + \epsilon \lambda^2 \frac{z'}{\lambda^2 z \log(\lambda)} \quad (5.5)$$

where z' is the derivative of z at ϵ equal zero yielded by the perturbative solution of the system

$$\begin{cases} \sum_{p,q} [I_{m,m;p,q}^{(4;0)} + \epsilon I_{m,m;p,q}^{(4;1)}] C_{p,q}^{(4)}(\epsilon) = 0 \\ \sum_{p,q} [I_{m,m-1;p,q}^{(4;0)} + \epsilon I_{m,m-1;p,q}^{(4;1)}] C_{p,q}^{(4)}(\epsilon) = 0. \end{cases} \quad (5.6)$$

The correction to ρ_4 due to time correlation increases the anomaly leading to a slower decay of the diagonal moment. For z' is negative (see Fig. 3) the overall anomaly is

$$\rho_4(\epsilon) = (2 - \xi) - \frac{\log(z)}{\log(\lambda)} + \left| \epsilon \lambda^2 \frac{z'}{\lambda^2 z \log(\lambda)} \right| \quad (5.7)$$

In the range of reliability of first order perturbation theory the effect is very small: for $\epsilon \lambda^2 \approx O(10^{-1})$ the prediction is a correction amounting to the three percent of the white noise exponent $H(4)$. The perturbative scheme just proposed does not take into account the emergence of pure second neighbours interactions. In order to weight their relevance for the diagonal scaling and simultaneously to check the hypothesis of normal scaling for the marginal scaling in (5.4) one can relax the closure in order to encompass the equation

$$\sum_{p,q} [I_{m,m-2;p,q}^{(4;0)} + \epsilon I_{m,m-2;p,q}^{(4;1)}] C_{p,q}^{(4)}(\epsilon) = 0 \quad (5.8)$$

which describe the independent, in the sense stated above, second neighbours interaction. Consistency with the white noise theory imposes the latter equation to decouple when ϵ is set to zero. The requirement is satisfied if the closure is chosen in the form

$$C_{m+n, m}^{(4)} = x(\epsilon) q(\epsilon)^{\frac{(n-1)(n-2)}{2}} k_{(n-1)-H(2)} C_{m,m}^{(4)} \quad (5.9)$$

the pre-factor $q(\epsilon)^{\frac{(n-1)(n-2)}{2}}$ enforces $q(0)$ to be a function of the white noise renormalisation constants. Were the white noise closure exact it would fix the value of $q(0)$ to one.

In Fig. 4 the $q(0)$ is plotted versus ξ : through all the physical range it stays close to one with a maximal deviation on the order of four percent for ξ equal to minus two. Moreover as shown in Fig. 3 the time correlation induced correction to $H(4, \epsilon)$ when (5.8) is included has the same qualitative behaviour and is quantitatively very close to

the nearest-neighbours prediction. The result is an a-posteriori check of the robustness of the closure approach. It confirms that first order corrections can be extracted within the logical scheme of the zero order one. It follows that the equations specifying the zero modes of the inertial operator acting on the sixth moment (see appendix D) can be closed by assuming:

$$\begin{aligned}
C_{m+n,m+n,m+n}^{(6)} &= z(\epsilon)^{-l} C_{m,m,m}^{(6)} \\
C_{m+n,m+n,m}^{(6)} &= x_1(\epsilon) k_{n-1}^{-H(4,\epsilon)} C_{m,m,m,m}^{(6)} \\
C_{m+n+p,m+n,m}^{(6)} &= x_3(\epsilon) k_{p-1}^{-H(2)} k_{n-1}^{-H(4,\epsilon)} C_{m,m,m,m}^{(6)} \\
C_{m+n,m,m,m}^{(6)} &= x_2(\epsilon) k_{n-1}^{-H(2)} C_{m,m,m,m}^{(6)}.
\end{aligned} \tag{5.10}$$

The exponent $H(4, \epsilon)$ is known perturbatively from (5.5) while $H(2)$ is universal. With the same rationale (appendix D) one can evaluate $H(8, \epsilon)$.

In Fig. 5 the analytic predictions for the corrections to the scaling exponents are summarised. In all cases the corrections are negative i.e. they carry a positive contribution to $\rho_{2\omega}$. The corrections increase with ω , the rate of the growth being slightly slower than the $\Delta(2\omega) \propto \omega(\omega - 1) \Delta(4)/2$ predicted in [14] for time-correlation generalised PDE Kraichnan models.

Within the range of first order perturbation theory the overall effect of time correlation is seen to enhance intermittency. An intuitive understanding of the phenomenon might be obtained by interpreting the time correlation as a mechanism to increase the probability of coherent fluctuations of the scalar field. The latter are rare events felt in the tail of the probability density of the scalar field as extreme deviations from the Gaussian behaviour of the typical events.

VI. NUMERICAL EXPERIMENTS

The resort to numerical experiments has a double motivation. On one hand they can be used to test the predictions from the first order perturbation theory. On the other hand they provide a broader scenario of the features of the model beyond the grasp of perturbative approaches. The first task is far from being easy because a quantitative check of perturbation theory requires measurements of the scaling exponents within an accuracy smaller than two percent.

The main feature of the inertial range is the conservation of the scalar “energy”. From the analytical point of view this is seen in the non-commutativity of the terms associated with the multiplicative noise, B^γ in (2.10). This property rules out the use of a simple Euler scheme, which can be applied in the case of delta correlated noise. In the case of white noise advection the multiplicative structure of the noise (2.10), which is interpreted in the Stratonovich sense, can be mapped into the corresponding Itô equations. The advantage is that the diagonal non zero average part of the noise is explicitly turned into an effective drift term [26]. The non-diagonal terms in the Taylor expansion of the scalar field Θ are of the order three halves in dt , which are neglected in the Euler scheme. This procedure becomes meaningless for a time-correlated noise. There ordinary calculus holds and in the Taylor expansion of Θ both diagonal and non-diagonal products of the noise are of the same order in dt .

Moreover the algorithm to be used must tend smoothly to a white noise limit, so that the same relative error is preserved independently on the value of ϵ .

Following Burrage & Burrage [27] a reliable way out from the mentioned difficulties is to adopt the Trotter-Lie-Magnus formula to integrate the equations of motion to first order. For each time increment dt (2.10) is solved in exponential form. Fast matrix exponentiation algorithms are provided by the package EXPOKIT [28].

To generate the correlated noise, the exact method described by Miguel & Toral [29] is used. This method ensures that the noise is accurate down to the limit $\epsilon \rightarrow 0$.

The relevant time scale to measure the convergence of the solution is the slowest time scale in the system, namely the eddy-turn-over time of the first shell estimated as the maximum between $\epsilon\tau_1$ and τ_1 . As shown in Fig. 6 more than $N_\tau = 10000$ eddy-turn-over times are needed to achieve a converged solution for the sixth order structure function. The time step is set by the fastest time scale in the system, which is the one of the largest shell $\epsilon\tau_M$. The number of iterations needed to achieve convergence is then for ϵ less than one:

$$\#(\text{iterations}) = \frac{N_\tau \tau_M}{\epsilon \tau_1} = \frac{N_\tau}{\epsilon} \lambda^{(M-1)(1-\frac{\xi}{2})} \tag{6.1}$$

which shows that the number of iterations needed grows like $1/\epsilon$, making it difficult to get close to the white noise limit using the same algorithm.

The scaling of the diagonal moments of higher order has been extracted by means of extended self-similarity [30], where the p -th order structure function is plotted versus the second order one, which is assumed to be normal. The scaling is found as the average slope of the logarithmic derivatives in the inertial range.

We considered a system with twenty-five shells with wave numbers increasing as power of $\lambda = 2$, with viscosity $\kappa = 5 \times 10^{-9}$. This choice ensures that there are several shells in the dissipative range. We focused on the results for ξ equal two thirds (Kolmogorov scaling).

In Fig. 7 the “normalised” structure functions $\langle |\theta_m^p| \rangle k_m^{H(p)}$ are shown. The quality of the scaling is demonstrated by the fact that the moments show scaling over a wide range of scales.

A summary of the numerical experiments is given in Fig. 8 where the scaling exponents are plotted versus the order of the moments of the scalar field for different values of ϵ . It is evident that the anomaly grows as the time correlation increases.

When turning to the interpretation of the results in more detail, the uncertainty in the extraction of the scaling has to be kept in mind. For the sixth moment this uncertainty turned out to be on the order of a four percent. The changes in the scaling between different values of ϵ is also on the order of a few percent. This seems to exclude a proper resolution in the numerics to compare the results with the analytical predictions from the perturbation analysis. However, the results for different values of ϵ can still be compared with some confidence, as the relative uncertainty between the different runs is much smaller than the absolute uncertainty. This means that the slope of, i.e., the sixth order structure functions vs. ϵ will be well resolved, while the absolute values can be shifted up and down a few percent.

In Fig. 9 the analytical prediction of the exponents is compared with the result of the numerics. The theoretical points are systematically below the numerical ones which is due to the absolute uncertainty as explained above. The slope is the same for the analytical calculation and the numerics, giving credibility to the results of the perturbation analysis. It should be noted that the effect of time correlation on the anomaly is quantitatively quite small even in the non perturbative range when ϵ is equal to one ($\epsilon\lambda^2$ equal four).

The global picture provided by the numerical experiments is that $H(2\omega)$ is seen to be a non linear function of ϵ which, after rapid initial decrease in the perturbative range, displays a much slower rate of variation. An interesting question is whether there is a limiting value of the scaling of the structure function as $\epsilon \gg 1$ or not. However the quality of the numerics does not allow us to answer this question.

VII. CONCLUSION

We have presented a shell model for the advection of a passive scalar by a velocity field which is exponentially correlated in time. We developed a systematic procedure to calculate the exponents of the correlation of the diagonal moments (the structure functions). For the delta correlated velocity we find good agreement between analytical and numerical calculations up to the eight order. We presented an analytical perturbative theory to compute the correction to the scaling exponents due to the exponentially correlated velocity field.

The occurrence of anomalies in the exponents of the diagonal moments of the scalar and their non universality versus the intensity $\epsilon\lambda^2$ of the time correlation, is related to the presence of pure short range couplings in the corresponding inertial operator which provide for non trivial scaling of the zero modes. In the absence of such short range couplings, as is the case for the second moment, normal scaling would take place independently on the value of $\epsilon\lambda^2$.

The behaviour of the anomalous exponents in the non-perturbative regime was studied numerically. This was found to be a non linear monotonic function of $\epsilon\lambda^2$, decreasing with a rate much slower than in the perturbative regime. It is thus clear that the addition of the time correlation to the advecting velocity field enhances the anomalous scaling. The anomaly found in the present study is still much smaller than what is found when the passive scalar is driven by a turbulent velocity field driven by Navier-Stokes turbulence or by a shell model for the velocity field [9]. This indicates that the non-Gaussian nature of the real turbulent velocity field plays a significant rôle for the strong anomalous scaling observed for real passive scalars.

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APPENDIX A: STOCHASTIC INTEGRATION BY PARTS FORMULA

A heuristic proof of the stochastic integration by parts formula is provided. For a rigorous treatment see [18], [19]. Let ζ_t be a stochastic process whose realisations are defined as the solution of the Itô SDE

$$\dot{x}_t = b(x_t, t) + \sigma(x_t, t)\eta_t, \quad x_t|_{t=0} = x_0 \quad (\text{A1})$$

where η_t is white noise. Let ζ_t^ϵ the stochastic process specified by

$$\dot{x}_t = b(x_t, t) + \epsilon h(x_t, t)\sigma(x_t, t) + \sigma(x_t, t)\eta_t \quad x_t|_{t=0} = x_0 \quad (\text{A2})$$

For ϵ equal the two SDE's coalesce: (A2) can be derived from (A1) under the variation of the white noise $\eta_t \rightarrow \eta_t + h(x_t, t)$. The integration by parts formula states that for any smooth functional f the identity holds:

$$\left\langle \frac{d}{d\epsilon} f(\zeta_t^\epsilon) \right\rangle_{\zeta_t^\epsilon} |_{\epsilon=0} = \left\langle f(\zeta_t) \int_0^t ds h(\zeta_s, s) \right\rangle_{\zeta_t} \quad (\text{A3})$$

where $\langle \dots \rangle_{\zeta_t}$ denotes the expectation values with respect to the measure induced by ζ_t . In order to prove it let us observe that the transition probability density for (A2) can be written formally as a path integral (Itô discretisation):

$$p_{\eta^\epsilon}(x, t | x_0, 0) = \int_{x_0}^{x_t=x} \mathcal{D}x e^{-S_\zeta(x, t | x_0, 0) + \int_0^t dt' [\frac{x_{t'} - b(x_{t'})}{\sigma(x_{t'}, t')} \epsilon h(x_{t'}, t') - \frac{\epsilon^2}{2} h^2(x_{t'}, t')]} \\ S_\zeta(x, t | x_0, 0) = \int_0^t \frac{dt'}{2} [\frac{x_{t'} - b(x_{t'})}{\sigma(x_{t'}, t')}]^2 \quad (\text{A4})$$

If one introduces the functional

$$M(\zeta_t^\epsilon) = e^{-\int_0^t dt' [\frac{x_{t'} - b(x_{t'})}{\sigma(x_{t'}, t')} \epsilon h(x_{t'}, t') - \frac{\epsilon^2}{2} h^2(x_{t'}, t')]} \quad (\text{A5})$$

one has by construction

$$\frac{d}{d\epsilon} \langle M(\zeta_t^\epsilon) f(\zeta_t^\epsilon) \rangle_{\zeta_t^\epsilon} = 0 \quad (\text{A6})$$

To each realisation of the solutions of (A2) corresponds a mapping $\eta_t \rightarrow x_t = x(t, \eta_t, \epsilon)$. Hence the last equality can be rewritten as the white noise average:

$$\frac{d}{d\epsilon} \langle M(x(t, \eta_t, \epsilon)) f(x(t, \eta_t, \epsilon)) \rangle_{\eta_t} = 0 \quad (\text{A7})$$

which implies (A3) when ϵ is set to zero. The derivative

$$\frac{d}{d\epsilon} f(\zeta_t^\epsilon) |_{\epsilon=0} = D\zeta_t \partial_{\zeta_t} f(\zeta_t) \quad (\text{A8})$$

is a Fréchet derivative. The dynamics of the stochastic process $D\zeta_t$ is linear once the realisations x_t of ζ_t are known:

$$y_t \equiv Dx_t \\ \dot{y}_t = y_t \partial_{x_t} [b(x_t, t) + \sigma(x_t, t)\eta_t] + h(x_t, t)\sigma(x_t, t) \quad (\text{A9})$$

It is worth to note that for $b = 0$, $\sigma = h = 1$ the integration by parts formula (A3) reduces to

$$t \langle \partial_{w_t} f(w_t) \rangle = \langle f(w_t) w_t \rangle \quad (\text{A10})$$

which is the Gaussian integration by parts formula (see e.g. [1]) applied to the Wiener process $\mathcal{N}(0, t)$.

The generalisation to a multidimensional complex case proceeds straightforwardly by introducing $2N$ variational parameters $\{\epsilon_i, \epsilon_i^*\}_{i=1}^{2N}$ and applying the definitions

$$\langle \eta_m(t) \eta_n^*(s) \rangle = 2 \delta_{mn} \delta(t - s) \quad (\text{A11})$$

for the white noise correlations.

APPENDIX B: STOCHASTIC INTEGRATION BY PARTS FOR THE O-U PROCESS

As in the above appendix we limit ourselves to the real case the generalisation to the complex case being straightforward. Functional differentiation is formally derived from a Fréchet derivative with $h(x_t, t) = \delta(t - s)$ where s is a parameter specifying the time when the white noise is perturbed. The variation is assumed to be non-anticipating (causal):

$$\lim_{s \uparrow t} \int_0^t ds' \delta(s - s') = 0 \quad (\text{B1})$$

Let us consider the system of SDE's

$$\dot{x}_m = b_m(x) + \sum_{n=1}^2 \sigma_{m,n}(x) c_n(t) \quad (\text{B2})$$

where c is the coloured noise:

$$c_n(t) = \int_0^t ds' \frac{e^{-\frac{t-s'}{\epsilon\tau_n}}}{\epsilon\sqrt{\tau_n}} \eta_n(s') \quad (\text{B3})$$

Functional differentiation gives

$$\frac{d}{dt}(D_l^s x_m) = \sum_{k=1}^N D_l^s x_k [\partial_k b_m(x) + \partial_k \sum_{n=1}^2 \sigma_{m,n}(x) \int_0^t ds' \frac{e^{-\frac{t-s'}{\epsilon\tau_n}}}{\epsilon\sqrt{\tau_n}} \eta_n(s')] + \frac{e^{-\frac{t-s}{\epsilon\tau_l}}}{\epsilon\sqrt{\tau_l}} \sigma_{m,l}(x) \quad (\text{B4})$$

The functional derivative is fully specified when it is known its form at the time s when the variation of the white noise occurs. The latter is determined by the causality requirement

$$\frac{d}{dt}(D_l^s c_n(t)) = [\partial_t \frac{e^{-\frac{t-s}{\epsilon\tau_n}}}{\epsilon\sqrt{\tau_n}} + \frac{1}{\epsilon\sqrt{\tau_n}} \delta(t - s)] \delta_{n,l} \quad (\text{B5})$$

which implies the variation of the coloured noise to be nonzero *only* immediately after the instantaneous kick

$$D_l^s c_n(t) = \frac{e^{-\frac{t-s}{\epsilon\tau_n}}}{\epsilon\sqrt{\tau_n}} \delta_{n,l} \quad \forall t \geq s \quad (\text{B6})$$

By differentiating (B4) one finds

$$\frac{d^2}{dt^2}(D_l^s x_m) = \frac{e^{-\frac{t-s}{\epsilon\tau_l}}}{\epsilon\sqrt{\tau_l}} \sigma_{m,l}(x) \delta(t - s) + \text{smooth terms} \quad (\text{B7})$$

From the last equation it stems that for $t = s$

$$\frac{d}{dt}(D_l^s x_m)|_{t=s} = \frac{1}{\epsilon\sqrt{\tau_n}} \sigma_{m,l}(x) \quad (\text{B8})$$

Consistency with (B4) then requires that the variation of the x 's associated with a non anticipating variation of the white noise at time s fulfils the initial condition:

$$D_l^s x_m(s)|_{t=s} = 0 \quad (\text{B9})$$

The integration by parts formula (A3) for a smooth functional $O(x)$

$$\langle O(x_t) c_n(t) \rangle = \int_0^t ds' \frac{e^{-\frac{t-s'}{\epsilon\tau_n}}}{\epsilon\sqrt{\tau_n}} \sum_{l=1}^N \langle D^{s'} x_l \partial_{x_l} O(x_t) \rangle \quad (\text{B10})$$

The variation is the solution of the linear problem (B4) of which we define R to be the fundamental solution. It follows

$$\langle O(x_t) c_n(t) \rangle = \int_0^t ds \frac{e^{-\frac{t-s}{\epsilon\tau_n}}}{\epsilon} \int_s^t ds' \frac{e^{-\frac{s'-s}{\epsilon\tau_n}}}{\epsilon} \sum_{l=1}^N \sum_{m=1}^N \langle [\partial_{x_l} O(x_t)] R_{l,m}(t, s') \sigma_{m,n}(x'_s) \rangle \quad (\text{B11})$$

Finally inverting the order of integration one obtains

$$\langle O(x_t) c_n(t) \rangle = \int_0^t ds' \frac{e^{-\frac{t-s'}{\epsilon\tau_n}} - e^{-\frac{t+s'}{\epsilon\tau_n}}}{2\epsilon} \sum_{l=1}^N \sum_{m=1}^N \langle [\partial_{x_l} O(x_t)] R_{l,m}(t, s') \sigma_{m,n}(x'_s) \rangle \quad (\text{B12})$$

This proves the real version of formula (3.1).

APPENDIX C: THE FOURTH ORDER CORRELATION TO FIRST ORDER

The inertial operator acting on the fourth moment $C_{m,n}^{(4)}(t)$ is in the large time limit

$$\begin{aligned} RHS &= I_{m,n;p,q}^{(4,0)} C_{p,q}^{(4)} - 2k_{m+1}^2 d_m \int_0^t ds e^{-\frac{t-s}{\epsilon\tau_m}} \frac{d}{ds} \Re \mathcal{F}_{m,n}^{(4)}(t, s) + \\ &+ k_m^2 d_{m-1} \int_0^t ds e^{-\frac{t-s}{\epsilon\tau_{m-1}}} \frac{d}{ds} \Re \mathcal{F}_{m-1,n}^{(4)}(t, s) + \\ &- k_{n+1}^2 d_n \int_0^t ds e^{-\frac{t-s}{\epsilon\tau_n}} \frac{d}{ds} \Re \mathcal{F}_{n,m}^{(4)}(t, s) \\ &+ k_n^2 d_{n-1} \int_0^t ds e^{-\frac{t-s}{\epsilon\tau_{n-1}}} \frac{d}{ds} \Re \mathcal{F}_{n-1,m}^{(4)}(t, s) \end{aligned} \quad (\text{C1})$$

The bi-dimensional matrix $I_{m,n;p,q}^{(4,0)}$ is the white noise linear inertial operator. The corrections to the white noise theory are generated by the time derivative at equal times of the integrand function $\Re \mathcal{F}_{n,m}^{(4)}$

$$\begin{aligned} \mathcal{F}_{m,n}^{(4)}(t, s) &\doteq \langle \Theta_{N+m}(t) \Theta_{N+n}(t) \Theta_n(t) R_{N+m+1, N+m}(t, s) \Theta_{m+1}(s) \rangle + \\ &- \langle \Theta_{N+m}(t) \Theta_{N+n}(t) \Theta_n(t) R_{N+m+1, N+m+1}(t, s) \Theta_m(s) \rangle + \\ &+ \langle \Theta_{N+m+1}(t) \Theta_{N+n}(t) \Theta_n(t) R_{N+m, N+m}(t, s) \Theta_{m+1}(s) \rangle + \\ &- \langle \Theta_{N+m+1}(t) \Theta_{N+n}(t) \Theta_n(t) R_{N+m, N+m+1}(t, s) \Theta_m(s) \rangle + \\ &+ \langle \Theta_{N+m}(t) \Theta_{N+m+1}(t) \Theta_{N+n}(t) R_{n, N+m}(t, s) \Theta_{m+1}(s) \rangle + \\ &- \langle \Theta_{N+m}(t) \Theta_{N+m+1}(t) \Theta_{N+n}(t) R_{n, N+m+1}(t, s) \Theta_m(s) \rangle + \\ &+ \langle \Theta_{N+m}(t) \Theta_{N+m+1}(t) \Theta_n(t) R_{N+n, N+m}(t, s) \Theta_{m+1}(s) \rangle + \\ &- \langle \Theta_{N+m}(t) \Theta_{N+m+1}(t) \Theta_n(t) R_{N+n, N+m+1}(t, s) \Theta_m(s) \rangle \end{aligned} \quad (\text{C2})$$

After a double integration by parts neglecting viscous contributions one gets into

$$\begin{aligned} &\sum_{p,q} \frac{(I_{m,n;p,q}^{(4;0)} + \epsilon I_{m,n;p,q}^{(4;1)})}{2} C_{q,p}^{(4)} = \\ &= \left(-\frac{\lambda^2}{\tau_{-1+m}} - \frac{\lambda^2}{\tau_m} - \frac{\lambda^2}{\tau_n} - \frac{\lambda^2}{\tau_{-1+n}} + \frac{7\epsilon\lambda^4}{\tau_{-1+m}} + \frac{7\epsilon\lambda^4}{\tau_m} + \frac{7\epsilon\lambda^4}{\tau_{-1+n}} + \frac{7\epsilon\lambda^4}{\tau_n} \right) C_{m,n}^{(4)} + \\ &+ \left(\frac{\lambda^2}{\tau_m} - \frac{\epsilon\lambda^4}{\tau_{1+m}} - \frac{2\epsilon\lambda^4}{\tau_{-1+n}} - \frac{2\epsilon\lambda^4}{\tau_n} - \frac{7\epsilon\lambda^4}{\tau_m} \right) C_{m+1,n}^{(4)} + \\ &+ \left(\frac{\lambda^2}{\tau_n} - \frac{\epsilon\lambda^4}{\tau_{1+n}} - \frac{2\epsilon\lambda^4}{\tau_{-1+m}} - \frac{2\epsilon\lambda^4}{\tau_m} - \frac{7\epsilon\lambda^4}{\tau_n} \right) C_{m,n+1}^{(4)} + \\ &+ \left(\frac{\lambda^2}{\tau_{-1+m}} - \frac{\epsilon\lambda^4}{\tau_{-2+m}} - \frac{2\epsilon\lambda^4}{\tau_{-1+n}} - \frac{2\epsilon\lambda^4}{\tau_n} - \frac{7\epsilon\lambda^4}{\tau_{-1+m}} \right) C_{m-1,n}^{(4)} + \\ &+ \left(\frac{\lambda^2}{\tau_{-1+n}} - \frac{\epsilon\lambda^4}{\tau_{-2+n}} - \frac{2\epsilon\lambda^4}{\tau_{-1+m}} - \frac{2\epsilon\lambda^4}{\tau_m} - \frac{7\epsilon\lambda^4}{\tau_{-1+n}} \right) C_{m,n-1}^{(4)} + \end{aligned}$$

$$\begin{aligned}
& + \left(\frac{2\epsilon\lambda^4}{\tau_m} + \frac{2\epsilon\lambda^4}{\tau_n} \right) C_{m+1,n+1}^{(4)} + \left(\frac{2\epsilon\lambda^4}{\tau_{-1+m}} + \frac{2\epsilon\lambda^4}{\tau_{-1+n}} \right) C_{m-1,n-1}^{(4)} + \\
& + \left(\frac{2\epsilon\lambda^4}{\tau_m} + \frac{2\epsilon\lambda^4}{\tau_{-1+n}} \right) C_{m+1,n-1}^{(4)} + \left(\frac{2\epsilon\lambda^4}{\tau_{-1+m}} + \frac{2\epsilon\lambda^4}{\tau_n} \right) C_{m-1,n+1}^{(4)} + \\
& + \frac{\epsilon\lambda^4}{\tau_{-2+m}} C_{m-2,n}^{(4)} + \frac{\epsilon\lambda^4}{\tau_{1+m}} C_{m+2,n}^{(4)} + \frac{\epsilon\lambda^4}{\tau_{-2+n}} C_{m,n-2}^{(4)} + \frac{\epsilon\lambda^4}{\tau_{1+n}} C_{m,n+2}^{(4)} + \\
& + \delta_{n,m} \left[\left(\frac{2\lambda^2}{\tau_m} - \frac{2\epsilon\lambda^4}{\tau_{1+m}} - \frac{4\epsilon\lambda^4}{\tau_{-1+m}} - \frac{34\epsilon\lambda^4}{\tau_m} \right) C_{m+1,m}^{(4)} + \right. \\
& + \left(\frac{2\lambda^2}{\tau_{-1+m}} - \frac{2\epsilon\lambda^4}{\tau_{-2+m}} - \frac{4\epsilon\lambda^4}{\tau_m} - \frac{34\epsilon\lambda^4}{\tau_{-1+m}} \right) C_{m,m-1}^{(4)} + \\
& + \left(\frac{4\epsilon\lambda^4}{\tau_{-1+m}} + \frac{4\epsilon\lambda^4}{\tau_m} \right) C_{m,m}^{(4)} + \left(\frac{4\epsilon\lambda^4}{\tau_{-1+m}} + \frac{4\epsilon\lambda^4}{\tau_m} \right) C_{m+1,m-1}^{(4)} + \\
& + \frac{4\epsilon\lambda^4}{\tau_m} C_{m+1,m+1}^{(4)} + \frac{4\epsilon\lambda^4}{\tau_{-1+m}} C_{m-1,m-1}^{(4)} + \frac{2\epsilon\lambda^4}{\tau_{1+m}} C_{m+2,m}^{(4)} + \frac{2\epsilon\lambda^4}{\tau_{-2+m}} C_{m,m-2}^{(4)} \left. \right] + \\
& + \delta_{n,m+1} \left[\left(-\frac{2\lambda^2}{\tau_m} + \frac{3\epsilon\lambda^4}{\tau_{-1+m}} + \frac{3\epsilon\lambda^4}{\tau_{1+m}} + \frac{34\epsilon\lambda^4}{\tau_m} \right) C_{m+1,m}^{(4)} - \frac{4\epsilon\lambda^4}{\tau_m} C_{m,m}^{(4)} + \right. \\
& + \left(\frac{3\epsilon\lambda^4}{\tau_{-1+m}} + \frac{\epsilon\lambda^4}{\tau_m} \right) C_{m,m-1}^{(4)} - \left(\frac{3\epsilon\lambda^4}{\tau_{-1+m}} + \frac{\epsilon\lambda^4}{\tau_m} \right) C_{m+1,m-1}^{(4)} - \frac{4\epsilon\lambda^4}{\tau_m} C_{m+1,m+1}^{(4)} + \\
& + \left(\frac{\epsilon\lambda^4}{\tau_m} + \frac{3\epsilon\lambda^4}{\tau_{1+m}} \right) C_{m+2,m+1}^{(4)} - \left(\frac{\epsilon\lambda^4}{\tau_m} + \frac{3\epsilon\lambda^4}{\tau_{1+m}} \right) C_{m+2,m}^{(4)} \left. \right] + \\
& + \delta_{n,m-1} \left[\left(-\frac{2\lambda^2}{\tau_{-1+m}} + \frac{3\epsilon\lambda^4}{\tau_{-2+m}} + \frac{3\epsilon\lambda^4}{\tau_m} + \frac{34\epsilon\lambda^4}{\tau_{-1+m}} \right) C_{m,m-1}^{(4)} - \frac{4\epsilon\lambda^4}{\tau_{-1+m}} C_{m,m}^{(4)} + \right. \\
& + \left(\frac{\epsilon\lambda^4}{\tau_{-1+m}} + \frac{3\epsilon\lambda^4}{\tau_m} \right) C_{m+1,m}^{(4)} - \left(\frac{\epsilon\lambda^4}{\tau_{-1+m}} + \frac{3\epsilon\lambda^4}{\tau_m} \right) C_{m+1,m-1}^{(4)} - \frac{4\epsilon\lambda^4}{\tau_{-1+m}} C_{m-1,m-1}^{(4)} + \\
& + \left(\frac{3\epsilon\lambda^4}{\tau_{-2+m}} + \frac{\epsilon\lambda^4}{\tau_{-1+m}} \right) C_{m-1,m-2}^{(4)} - \left(\frac{3\epsilon\lambda^4}{\tau_{-2+m}} + \frac{\epsilon\lambda^4}{\tau_{-1+m}} \right) C_{m,m-2}^{(4)} \left. \right] + \\
& + \delta_{n,m+2} \left[\left(-\frac{3\epsilon\lambda^4}{\tau_m} - \frac{\epsilon\lambda^4}{\tau_{1+m}} \right) C_{m+1,m}^{(4)} \left(-\frac{\epsilon\lambda^4}{\tau_m} - \frac{3\epsilon\lambda^4}{\tau_{1+m}} \right) C_{m+2,m+1}^{(4)} + \right. \\
& + \left(\frac{\epsilon\lambda^4}{\tau_m} + \frac{\epsilon\lambda^4}{\tau_{1+m}} \right) C_{m+2,m}^{(4)} \left. \right] + \\
& + \delta_{n,m-2} \left[\left(-\frac{\epsilon\lambda^4}{\tau_{-2+m}} - \frac{3\epsilon\lambda^4}{\tau_{-1+m}} \right) C_{m,m-1}^{(4)} + \left(-\frac{3\epsilon\lambda^4}{\tau_{-2+m}} - \frac{\epsilon\lambda^4}{\tau_{-1+m}} \right) C_{m-1,m-2}^{(4)} + \right. \\
& + \left(\frac{\epsilon\lambda^4}{\tau_{-2+m}} + \frac{\epsilon\lambda^4}{\tau_{-1+m}} \right) C_{m,m-2}^{(4)} \left. \right]
\end{aligned} \tag{C3}$$

The diagonal scaling exponent is derived up to first order is ϵ resorting to linear perturbation theory. If pure second neighbours interactions are taken into account the constant $q(0)$ is specified by

$$q(0) = \frac{1 - (1 + \lambda^{-2H(2)} + \lambda^{-H(2)}) z(0)}{z(0) + z(0)^2 \lambda^{-3H(2)}} \tag{C4}$$

The result is approximately equal to one for all ξ ranging between $[0, 2]$. The first order correction $z'(0)$ is extracted from the solution of the linear system:

$$\begin{aligned}
& (4\lambda^{2+H(2)} + 4\lambda^2 z(0)) x'(0) + 4\lambda^2 x(0) z'(0) = \\
& = -4\lambda^{4+2H(2)} x(0) + 4\lambda^{4-H(2)} x(0) z(0) + 4\lambda^{2+2(1-H(2))} x(0) z(0)^2 + \\
& + 2\lambda^4 (9 - 4x(0) + (4 - 22x(0)) z(0)) + 2\lambda^{4+H(2)} (4 + (9 - 24x(0)) - 4x(0) z(0)) \\
& [\lambda^2 (-3 - \lambda^{-H(2)} - \lambda^{H(2)}) z(0) + \lambda^{2-2H(2)} z(0)^2] x'(0) + \\
& + [\lambda^2 + (-3 - \lambda^{-H(2)} - \lambda^{H(2)}) x(0) \lambda^2 + 2\lambda^{2-2H(2)} x(0) z(0)] z'(0) =
\end{aligned}$$

$$\begin{aligned}
&= -13\lambda^4 - 3\lambda^{4+H(2)} + 3\lambda^{4+H(2)}(2 + \lambda^{-H(2)})x(0) - 13\lambda^4 z(0) - 3\lambda^{4-H(2)}z(0) + \\
&+ \lambda^4(42 - 2\lambda^{-2H(2)} + 7\lambda^{-H(2)} + 9\lambda^{H(2)} + q(0))x(0)z(0) + \\
&+ \lambda^4 x(0)z(0)^2[1 - 10\lambda^{-2H(2)} + (-1 + 2q(0))\lambda^{-3H(2)} + (3 + 2q(0))\lambda^{-H(2)}] + \\
&+ \lambda^{4-4H(2)}q(0)x(0)z(0)^3 \\
&\lambda^{2-H(2)}x(0)[1 - 2z(0)(1 - q(0) + 3\lambda^{-2H(2)}q(0)z(0) + \lambda^{-2H(2)} + \lambda^{-H(2)})]z'(0) + \\
&+ \lambda^{2-H(2)}z(0)[1 - z(0)(1 + \lambda^{-2H(2)} + \lambda^{-H(2)} - q(0) + \lambda^{-2H(2)}q(0)z(0))]x'(0) + \\
&+ \lambda^{2-H(2)}z(0)^2[x(0) + \lambda^{-2H(2)}x(0)z(0)]q'(0) = \\
&= 2\lambda^{4+H(2)}(-1 + \lambda^{-3H(2)} - 4\lambda^{-2H(2)} - 2\lambda^{-H(2)})x(0)z(0) + \\
&+ \lambda^{4-H(2)}z(0)^2 + 6\lambda^{2+2(1-H(2))}x(0)z(0)^2 - 2\lambda^{4-4H(2)}q(0)x(0)z(0)^2 + \\
&- \lambda^4(1 + q(0))x(0)z(0)^2 - \lambda^{4-3H(2)}(-7 + 2q(0))x(0)z(0)^2 + \\
&+ \lambda^{4-H(2)}(2 - 7q(0) + q(0)^3)x(0)z(0)^2 + 2\lambda^{4-H(2)}(1 + \lambda^{-2H(2)})x(0)z(0)^3 + \\
&- \lambda^{4-H(2)}(2 + \lambda^{-4H(2)} + 7\lambda^{3H(2)} + \lambda^4 + 2\lambda^4(1 + \lambda^{-H(2)})z(0) + \\
&+ 2\lambda^{H(2)})q(0)x(0)z(0)^3 + 2\lambda^{4-2H(2)}(1 + \lambda^{-3H(2)})q(0)^3x(0)z(0)^3 + \\
&+ \lambda^{4-6H(2)}q(0)^3x(0)z(0)^4
\end{aligned} \tag{C5}$$

APPENDIX D: THE INERTIAL OPERATOR FOR THE SIXTH MOMENT OF THE CORRELATION UP TO FIRST ORDER

Under the hypothesis that pure second neighbours interactions do not require new equations to specify the diagonal scaling for small values of ϵ , there are only four equations describing how global coupling are “renormalised” by relevant pure short range interactions. Given the m -th shell one has

$$\begin{aligned}
0 &= \sum_{p,q,r} [I_{m,m,m;p,q,r}^{(6;0)} + \epsilon I_{m,m,m;p,q,r}^{(6;1)}] C_{p,q,r}^{(6);1} = \\
&= -\left(\frac{3\lambda^2}{\tau_{-1+m}} + \frac{3\lambda^2}{\tau_m} - \frac{45\epsilon\lambda^4}{\tau_{-1+m}} - \frac{45\epsilon\lambda^4}{\tau_m}\right) C_{m,m,m}^{(6)} + \\
&+ \left(\frac{9\lambda^2}{\tau_m} - \frac{9\epsilon\lambda^4}{\tau_{1+m}} - \frac{36\epsilon\lambda^4}{\tau_{-1+m}} - \frac{207\epsilon\lambda^4}{\tau_m}\right) C_{m,m,1+m}^{(6)} + \\
&+ \left(\frac{9\lambda^2}{\tau_{-1+m}} - \frac{9\epsilon\lambda^4}{\tau_{-2+m}} - \frac{36\epsilon\lambda^4}{\tau_m} - \frac{207\epsilon\lambda^4}{\tau_{-1+m}}\right) C_{m,m,-1+m}^{(6)} + \\
&+ \left(\frac{72\epsilon\lambda^4}{\tau_{-1+m}} + \frac{72\epsilon\lambda^4}{\tau_m}\right) C_{-1+m,m,1+m}^{(6)} + \frac{72\epsilon\lambda^4}{\tau_m} C_{1+m,1+m,m}^{(6)} + \\
&+ \frac{72\epsilon\lambda^4}{\tau_{-1+m}} C_{-1+m,-1+m,m}^{(6)} + \frac{9\epsilon\lambda^4}{\tau_{1+m}} C_{m,m,2+m}^{(6)} + \frac{9\epsilon\lambda^4}{\tau_{-2+m}} C_{m,m,-2+m}^{(6)} \\
0 &= \sum_{p,q,r} [I_{m,m,m-1;p,q,r}^{(6;0)} + \epsilon I_{m,m,m-1;p,q,r}^{(6;1)}] C_{p,q,r}^{(6)} = \\
&= \left(\frac{\lambda^2}{\tau_{-1+m}} - \frac{5\epsilon\lambda^4}{\tau_m} - \frac{23\epsilon\lambda^4}{\tau_{-1+m}}\right) C_{m,m,m}^{(6)} + \left(\frac{10\epsilon\lambda^4}{\tau_{-1+m}} + \frac{15\epsilon\lambda^4}{\tau_m}\right) C_{m,m,1+m}^{(6)} + \\
&- \left(\frac{\lambda^2}{\tau_{-2+m}} + \frac{2\lambda^2}{\tau_m} + \frac{7\lambda^2}{\tau_{-1+m}} - \frac{17\epsilon\lambda^4}{\tau_{-2+m}} - \frac{40\epsilon\lambda^4}{\tau_m} - \frac{169\epsilon\lambda^4}{\tau_{-1+m}}\right) C_{m,m,-1+m}^{(6)} + \\
&+ \left(\frac{4\lambda^2}{\tau_m} - \frac{4\epsilon\lambda^4}{\tau_{1+m}} - \frac{8\epsilon\lambda^4}{\tau_{-2+m}} - \frac{36\epsilon\lambda^4}{\tau_{-1+m}} - \frac{96\epsilon\lambda^4}{\tau_m}\right) C_{-1+m,m,1+m}^{(6)} + \\
&+ \left(\frac{4\lambda^2}{\tau_{-1+m}} - \frac{8\epsilon\lambda^4}{\tau_m} - \frac{12\epsilon\lambda^4}{\tau_{-2+m}} - \frac{124\epsilon\lambda^4}{\tau_{-1+m}}\right) C_{-1+m,-1+m,m}^{(6)} +
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{8\epsilon\lambda^4}{\tau_{-1+m}} + \frac{8\epsilon\lambda^4}{\tau_m} \right) C_{-1+m,-1+m,1+m}^{(6)} + \frac{8\epsilon\lambda^4}{\tau_m} C_{1+m,1+m,-1+m}^{(6)} + \\
& + \frac{8\epsilon\lambda^4}{\tau_{-1+m}} C_{-1+m,-1+m,-1+m}^{(6)} + \left(\frac{12\epsilon\lambda^4}{\tau_{-1+m}} + \frac{24\epsilon\lambda^4}{\tau_{-2+m}} \right) C_{-2+m,-1+m,m}^{(6)} + \\
& + \left(\frac{\lambda^2}{\tau_{-2+m}} - \frac{\epsilon\lambda^4}{\tau_{-3+m}} - \frac{4\epsilon\lambda^4}{\tau_m} - \frac{6\epsilon\lambda^4}{\tau_{-1+m}} - \frac{17\epsilon\lambda^4}{\tau_{-2+m}} \right) C_{m,m,-2+m}^{(6)} + \\
& + \left(\frac{8\epsilon\lambda^4}{\tau_{-2+m}} + \frac{8\epsilon\lambda^4}{\tau_m} \right) C_{-2+m,m,1+m}^{(6)} + \frac{4\epsilon\lambda^4}{\tau_{1+m}} C_{-1+m,m,2+m}^{(6)} + \frac{\epsilon\lambda^4}{\tau_{-3+m}} C_{m,m,-3+m}^{(6)}
\end{aligned}$$

$$\begin{aligned}
& \sum_{p,q,r} [I_{m,m-1,m-1;p,q,r}^{(6;0)} + \epsilon I_{m,m-1,m-1;p,q,r}^{(6;1)}] C_{p,q,r}^{(6)} = \\
& = \frac{8\epsilon\lambda^4}{\tau_{-1+m}} C_{m,m,m}^{(6)} + \left(\frac{4\lambda^2}{\tau_{-1+m}} - \frac{8\epsilon\lambda^4}{\tau_{-2+m}} - \frac{12\epsilon\lambda^4}{\tau_m} - \frac{124\epsilon\lambda^4}{\tau_{-1+m}} \right) C_{m,m,-1+m}^{(6)} + \\
& + \left(\frac{12\epsilon\lambda^4}{\tau_{-1+m}} + \frac{24\epsilon\lambda^4}{\tau_m} \right) C_{-1+m,m,1+m}^{(6)} + \\
& - \left(\frac{\lambda^2}{\tau_m} + \frac{2\lambda^2}{\tau_{-2+m}} + \frac{7\lambda^2}{\tau_{-1+m}} - \frac{17\epsilon\lambda^4}{\tau_m} - \frac{40\epsilon\lambda^4}{\tau_{-2+m}} - \frac{169\epsilon\lambda^4}{\tau_{-1+m}} \right) C_{-1+m,-1+m,m}^{(6)} + \\
& + \left(\frac{\lambda^2}{\tau_m} - \frac{4\epsilon\lambda^4}{\tau_{-2+m}} - \frac{6\epsilon\lambda^4}{\tau_{-1+m}} - \frac{17\epsilon\lambda^4}{\tau_m} - \frac{\epsilon\lambda^4}{\tau_{1+m}} \right) C_{-1+m,-1+m,1+m}^{(6)} + \\
& + \left(\frac{\lambda^2}{\tau_{-1+m}} - \frac{5\epsilon\lambda^4}{\tau_{-2+m}} - \frac{23\epsilon\lambda^4}{\tau_{-1+m}} \right) C_{-1+m,-1+m,-1+m}^{(6)} + \\
& + \left(\frac{8\epsilon\lambda^4}{\tau_{-2+m}} + \frac{8\epsilon\lambda^4}{\tau_{-1+m}} \right) C_{m,m,-2+m}^{(6)} + \left(\frac{15\epsilon\lambda^4}{\tau_{-2+m}} + \frac{10\epsilon\lambda^4}{\tau_{-1+m}} \right) C_{-1+m,-1+m,-2+m}^{(6)} + \\
& + \left(\frac{4\lambda^2}{\tau_{-2+m}} - \frac{4\epsilon\lambda^4}{\tau_{-3+m}} - \frac{8\epsilon\lambda^4}{\tau_m} - \frac{36\epsilon\lambda^4}{\tau_{-1+m}} - \frac{96\epsilon\lambda^4}{\tau_{-2+m}} \right) C_{-2+m,-1+m,m}^{(6)} + \\
& + \frac{\epsilon\lambda^4}{\tau_{1+m}} C_{-1+m,-1+m,2+m}^{(6)} + \left(\frac{8\epsilon\lambda^4}{\tau_{-2+m}} + \frac{8\epsilon\lambda^4}{\tau_m} \right) C_{-2+m,-1+m,1+m}^{(6)} + \\
& + \frac{8\epsilon\lambda^4}{\tau_{-2+m}} C_{-2+m,-2+m,m}^{(6)} + \frac{4\epsilon\lambda^4}{\tau_{-3+m}} C_{-3+m,-1+m,m}^{(6)}
\end{aligned}$$

$$\begin{aligned}
& \sum_{p,q,r} [I_{m,m+1,m-1;p,q,r}^{(6;0)} + \epsilon I_{m,m+1,m-1;p,q,r}^{(6;1)}] C_{p,q,r}^{(6)} = \\
& = \left(\frac{2\epsilon\lambda^4}{\tau_{-1+m}} + \frac{2\epsilon\lambda^4}{\tau_m} \right) C_{m,m,m}^{(6)} + \left(\frac{\lambda^2}{\tau_{-1+m}} - \frac{2\epsilon\lambda^4}{\tau_{1+m}} - \frac{12\epsilon\lambda^4}{\tau_m} - \frac{22\epsilon\lambda^4}{\tau_{-1+m}} \right) C_{m,m,1+m}^{(6)} + \\
& + \left(\frac{\lambda^2}{\tau_m} - \frac{2\epsilon\lambda^4}{\tau_{-2+m}} - \frac{12\epsilon\lambda^4}{\tau_{-1+m}} - \frac{22\epsilon\lambda^4}{\tau_m} \right) C_{m,m,-1+m}^{(6)} + \\
& + \left(\frac{3\epsilon\lambda^4}{\tau_{-1+m}} + \frac{6\epsilon\lambda^4}{\tau_m} \right) C_{1+m,1+m,m}^{(6)} + \left(\frac{3\epsilon\lambda^4}{\tau_m} + \frac{6\epsilon\lambda^4}{\tau_{-1+m}} \right) C_{-1+m,-1+m,m}^{(6)} + \\
& - \left(\frac{\lambda^2}{\tau_{-2+m}} + \frac{\lambda^2}{\tau_{1+m}} + \frac{4\lambda^2}{\tau_{-1+m}} + \frac{4\lambda^2}{\tau_m} - \frac{18\epsilon\lambda^4}{\tau_{-2+m}} - \frac{18\epsilon\lambda^4}{\tau_{1+m}} - \frac{84\epsilon\lambda^4}{\tau_{-1+m}} - \frac{84\epsilon\lambda^4}{\tau_m} \right) C_{-1+m,m,1+m}^{(6)} + \\
& + \left(\frac{\lambda^2}{\tau_m} - \frac{2\epsilon\lambda^4}{\tau_{-2+m}} - \frac{3\epsilon\lambda^4}{\tau_{-1+m}} - \frac{3\epsilon\lambda^4}{\tau_{1+m}} - \frac{20\epsilon\lambda^4}{\tau_m} \right) C_{1+m,1+m,-1+m}^{(6)} + \\
& + \left(\frac{\lambda^2}{\tau_{-1+m}} - \frac{2\epsilon\lambda^4}{\tau_{1+m}} - \frac{3\epsilon\lambda^4}{\tau_{-2+m}} - \frac{3\epsilon\lambda^4}{\tau_m} - \frac{20\epsilon\lambda^4}{\tau_{-1+m}} \right) C_{-1+m,-1+m,1+m}^{(6)} + \\
& + \left(\frac{3\epsilon\lambda^4}{\tau_m} + \frac{6\epsilon\lambda^4}{\tau_{1+m}} \right) C_{-1+m,1+m,2+m}^{(6)} + \left(\frac{3\epsilon\lambda^4}{\tau_{-1+m}} + \frac{6\epsilon\lambda^4}{\tau_{-2+m}} \right) C_{-2+m,-1+m,1+m}^{(6)} + \\
& + \left(\frac{2\epsilon\lambda^4}{\tau_{-1+m}} + \frac{2\epsilon\lambda^4}{\tau_{1+m}} \right) C_{-1+m,-1+m,2+m}^{(6)} + \left(\frac{2\epsilon\lambda^4}{\tau_{-1+m}} + \frac{2\epsilon\lambda^4}{\tau_{1+m}} \right) C_{m,m,2+m}^{(6)} +
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{2\epsilon\lambda^4}{\tau_{-2+m}} + \frac{2\epsilon\lambda^4}{\tau_m} \right) C_{m,m,-2+m}^{(6)} + \left(\frac{2\epsilon\lambda^4}{\tau_{-2+m}} + \frac{2\epsilon\lambda^4}{\tau_m} \right) C_{1+m,1+m,-2+m}^{(6)} + \\
& + \left(\frac{\lambda^2}{\tau_{-2+m}} - \frac{\epsilon\lambda^4}{\tau_{-3+m}} - \frac{2\epsilon\lambda^4}{\tau_{1+m}} - \frac{3\epsilon\lambda^4}{\tau_{-1+m}} - \frac{8\epsilon\lambda^4}{\tau_m} - \frac{18\epsilon\lambda^4}{\tau_{-2+m}} \right) C_{-2+m,m,1+m}^{(6)} + \\
& + \left(\frac{\lambda^2}{\tau_{1+m}} - \frac{\epsilon\lambda^4}{\tau_{2+m}} - \frac{2\epsilon\lambda^4}{\tau_{-2+m}} - \frac{3\epsilon\lambda^4}{\tau_m} - \frac{8\epsilon\lambda^4}{\tau_{-1+m}} - \frac{18\epsilon\lambda^4}{\tau_{1+m}} \right) C_{-1+m,m,2+m}^{(6)} + \\
& + \left(\frac{2\epsilon\lambda^4}{\tau_{-2+m}} + \frac{2\epsilon\lambda^4}{\tau_{1+m}} \right) C_{-2+m,m,2+m}^{(6)} + \frac{\epsilon\lambda^4}{\tau_{2+m}} C_{-1+m,m,3+m}^{(6)} + \frac{\epsilon\lambda^4}{\tau_{-3+m}} C_{-3+m,m,1+m}^{(6)}
\end{aligned} \tag{D1}$$

APPENDIX E: THE INERTIAL OPERATOR FOR THE EIGHT MOMENT OF THE CORRELATION UP TO FIRST ORDER

The set of independent equations is finally given as:

$$\begin{cases}
\sum_{p,q,r,s} [I_{m,m,m;p,q,r,s}^{(8;0)} + \epsilon I_{m,m,m;p,q,r,s}^{(8;1)}] C_{p,q,r,s}^{(8)} = 0 \\
\sum_{p,q,r,s} [I_{m,m,m-1;p,q,r,s}^{(8;0)} + \epsilon I_{m,m,m-1;p,q,r,s}^{(8;1)}] C_{p,q,r,s}^{(8)} = 0 \\
\sum_{p,q,r,s} [I_{m,m,m-1,m-1;p,q,r,s}^{(8;0)} + \epsilon I_{m,m,m-1,m-1;p,q,r,s}^{(8;1)}] C_{p,q,r,s}^{(8)} = 0 \\
\sum_{p,q,r,s} [I_{m,m,m+1,m-1;p,q,r,s}^{(8;0)} + \epsilon I_{m,m,m+1,m-1;p,q,r,s}^{(8;1)}] C_{p,q,r,s}^{(8)} = 0 \\
\sum_{p,q,r,s} [I_{m,m-1,m-1,m-1;p,q,r,s}^{(8;0)} + \epsilon I_{m,m-1,m-1,m-1;p,q,r,s}^{(8;1)}] C_{p,q,r,s}^{(8)} = 0 \\
\sum_{p,q,r,s} [I_{m,m+1,m-1,m-1;p,q,r,s}^{(8;0)} + \epsilon I_{m,m+1,m-1,m-1;p,q,r,s}^{(8;1)}] C_{p,q,r,s}^{(8)} = 0 \\
\sum_{p,q,r,s} [I_{m,m+1,m+1,m-1;p,q,r,s}^{(8;0)} + \epsilon I_{m,m+1,m+1,m-1;p,q,r,s}^{(8;1)}] C_{p,q,r,s}^{(8)} = 0 \\
\sum_{p,q,r,s} [I_{m,m+1,m-1,m-2;p,q,r,s}^{(8;0)} + \epsilon I_{m,m+1,m-1,m-2;p,q,r,s}^{(8;1)}] C_{p,q,r,s}^{(8)} = 0
\end{cases} \tag{E1}$$

The closure is provided again assuming scaling for all the possible conditioned expectation values with respect to a given shell. It follows

$$\begin{aligned}
C_{m+n,m+n,m+n,m+n}^{(8)} &= z(\epsilon)^{-l} C_{m,m,m,m}^{(8)} \\
C_{m+n,m+n,m+n,m}^{(8)} &= y_1(\epsilon) k_{n-1}^{-H(6,\epsilon)} C_{m,m,m,m}^{(8)} \\
C_{m+n,m+n,m,m}^{(8)} &= y_2(\epsilon) k_{n-1}^{-H(4,\epsilon)} C_{m,m,m,m}^{(8)} \\
C_{m+n,m,m,m,m}^{(8)} &= y_3(\epsilon) k_{n-1}^{-H(2)} C_{m,m,m,m}^{(8)} \\
C_{m+n+p,m+n,m,m}^{(8)} &= y_4(\epsilon) k_{p-1}^{-H(2)} k_{n-1}^{-H(4,\epsilon)} C_{m,m,m,m}^{(8)} \\
C_{m+n+p,m+n,m+n,m}^{(8)} &= y_5(\epsilon) k_{p-1}^{-H(2)} k_{n-1}^{-H(6,\epsilon)} C_{m,m,m,m}^{(8)} \\
C_{m+n+p,m+n+p,m+n,m}^{(8)} &= y_6(\epsilon) k_{p-1}^{-H(4,\epsilon)} k_{n-1}^{-H(6,\epsilon)} C_{m,m,m,m}^{(8)} \\
C_{m+n+p+q,m+n+p,m+n,m}^{(8)} &= y_7(\epsilon) k_{q-1}^{-H(2)} k_{p-1}^{-H(4,\epsilon)} k_{n-1}^{-H(6,\epsilon)} C_{m,m,m,m}^{(8)}
\end{aligned} \tag{E2}$$

-
- [1] U.Frisch, *Turbulence: the legacy of A.N. Kolmogorov* (Cambridge University Press 1995)
 - [2] R.H. Kraichnan, Phys. Rev. Lett. **72**, 1016 (1994)
 - [3] D.Bernard, K.Gawedzki and A.Kupiainen Phys. Rev. **E 54** 2564 (1996)
 - [4] K. Gawedzki and A. Kupiainen, Phys. Rev. Lett. **75** 3834 (1995)
 - [5] M. Chertkov, G. Falkovich, I. Kolokolov and V. Lebedev, Phys. Rev. **E 52** 4924 (1995)
 - [6] M.Chertkov and G.Falkovich, Phys. Rev. Lett. **76** 2706 (1996)
 - [7] K.Gawedzki, chao-dyn/9803027
 - [8] U.Frisch, A.Mazzino and M.Vergassola, Phys. Rev. Lett. **80**, 5532 (1998) and cond-mat/9802192

- [9] M.H. Jensen, G. Paladin and A. Vulpiani, Phys. Rev. **A 45**, N. 10, 7214 (1992)
- [10] T.Bohr, M.H. Jensen, G.Paladin and A.Vulpiani *Dynamical Systems Approach to Turbulence* Cambridge University Press (1998)
- [11] A.M. Obukhov, Izv. Akad. SSSR, Serv Geogr. Geofiz. **13**, 58 (1949)
- [12] S. Corrsin, J.Appl. Phys. **22**, 469 (1951)
- [13] A.Erdelyi *Asymptotic expansions* (Dover Publications Inc. New York 1956)
- [14] M.Chertkov, G.Falkovich and V.Lebedev Phys.Rev.Lett. **76**, 3707 (1996)
- [15] L.Arnold: *Stochastic differential equations: Theory and Applications* (Wiley, New York 1974)
- [16] A.Wirth and L.Biferale, Phys.Rev. E **54**, 4982 (1996)
- [17] R. Benzi, L.Biferale and A.Wirth, Phys. Rev. Lett. **78**, p. 4926 (1997) .
- [18] R.F.Bass, *Diffusions and elliptic operators* New York Springer-Verlag 1998
- [19] D.Nualart, *The Malliavin calculus and related topics* New York London Springer-Verlag 1995
- [20] J.Cardy *Scaling and Renormalization in Statistical Physics* Cambridge Lecture Notes in Physics, Cambridge University Press (1996)
- [21] J Zinn-Justin *Quantum Field Theory and Critical Phenomena* Clarendon Press Oxford (1989)
- [22] L.F. Richardson *Weather prediction by Numerical Process* Cambridge University Press, Cambridge (1922)
- [23] G. Parisi and U.Frisch *Turbulence and Predictability in Geophysical Fluid Dynamics*, Proceed. Intern. School of Physics “E.Fermi” Varenna 1983 eds M.Ghil, R.Benzi and G.Parisi. North Holland, Amsterdam.
- [24] L.Biferale, G.Boffetta, A. Celani and F.Toschi *chao-dyn/9804035*
- [25] M.Le Bellac *Quantum and statistical field theory* Oxford University Press 1991
- [26] P.E.Kloeden and E.Platen *Numerical Solution of Stochastic Differential Equations* Springer-Verlag 1995
- [27] k.Burrage and P.Burrage preprint *High strong order methods for non commutative ordinary stochastic differential equations and the Magnus formula* <http://www.maths.uq.edu.au/~kb/papers.html>
- [28] R.B.Sidje, to appear in ACM-Transactions on Mathematical software
- [29] M.San Miguel and R.Toral in *Instabilities and Nonequilibrium Structures VI* E.Tirapegui and W.Zeller editors Kluwer Acad. Pub (1997) and cond-mat/9707147
- [30] R.Benzi, S.Ciliberto, R.Tripiccone, C Baudet and S.Succi Phys.Rev E. **48** R29 (1993)

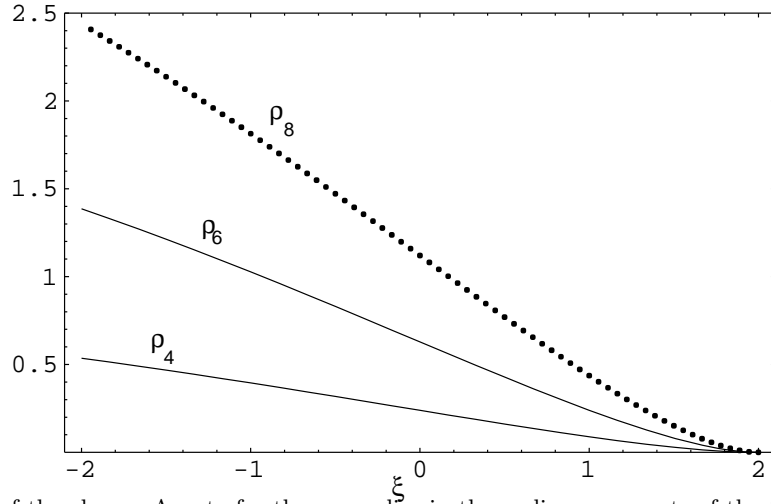


FIG. 1. The prediction of the closure Ansatz for the anomalies in the scaling exponents of the fourth, ρ_4 , the sixth, ρ_6 and the eight, ρ_8 , moments of the scalar field versus the turbulence parameter ξ . In all cases the anomalies are decreasing function of ξ going smoothly to zero as the Batchelor limit ξ equal to two is approached. The anomaly of the eight moment is obtained as the numerical solution of a sixth order polynomial.

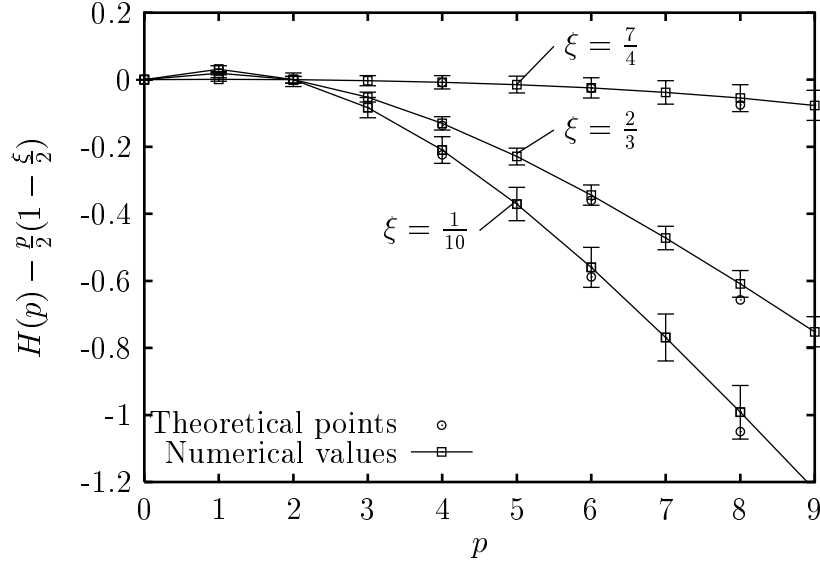


FIG. 2. The analytical prediction for the anomalous part of the scaling exponent compared with the result of the numerical experiments for different values of the turbulence degree parameter ξ . A Kolmogorov scaling of the advection field corresponds to $\xi = 2/3$. The dash-dotted line represents the (dimensional) normal scaling prediction. The continuum line interpolates the exponents as obtained from the numerical experiment (squares). The circles are the analytical prediction from the closure Ansatz.

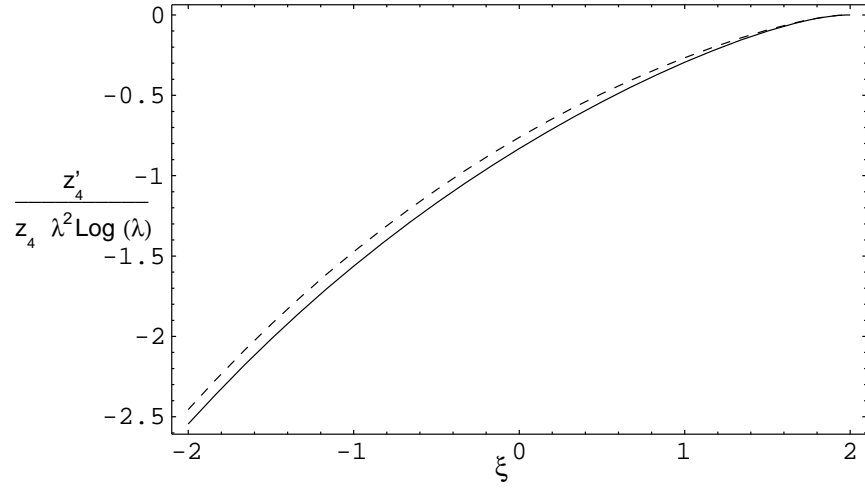


FIG. 3. the first order corrections to $H(4)$ with (dashed line) and without inclusion of second neighbours interactions are plotted versus the turbulence exponent. The inclusion of second neighbours couplings increases the intensity of the anomaly

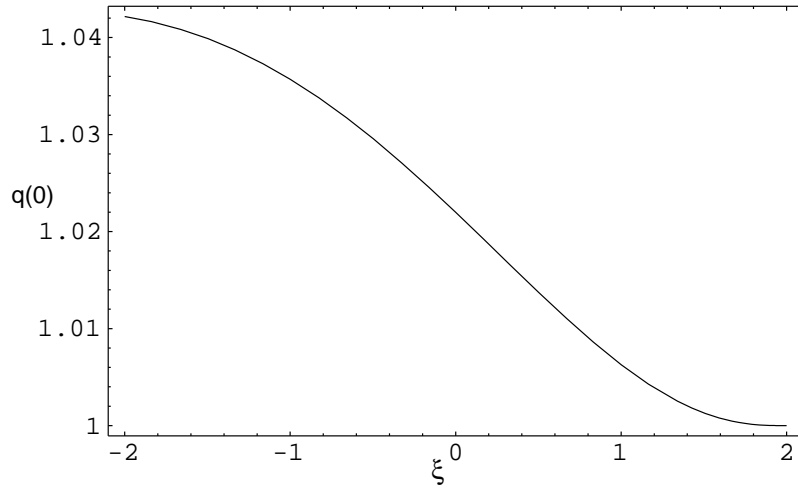


FIG. 4. The renormalisation constant $q(0)$ is plotted versus ξ . It remains close to one through all physical range proving self consistent the conjecture of normal scaling for the non-diagonal sector of the fourth moment. The result stresses that the renormalisation of nearest neighbours interactions provide the an accurate framework to extract the scaling exponent.

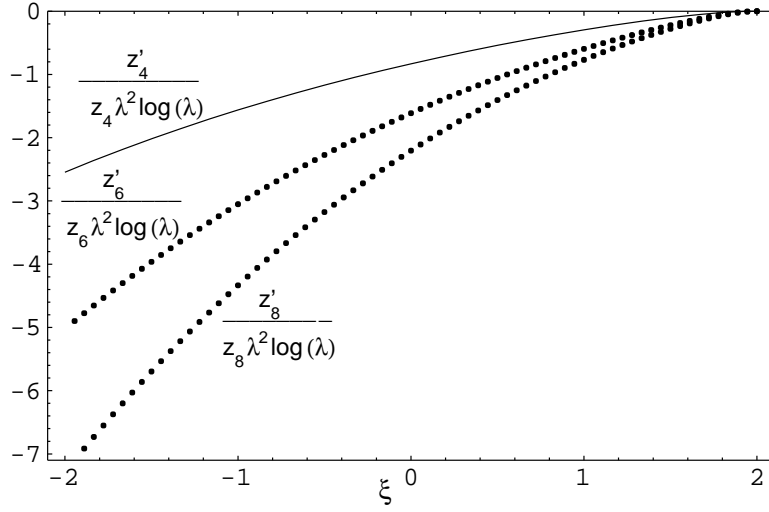


FIG. 5. The first order correction to $H(4)$ (continuous line), $H(6)$ and $H(8)$ versus the turbulence parameter ξ . In the last two cases the corresponding linear systems are solved numerically. In all cases the corrections are derived by perturbing the white noise closure “renormalisation” constants. The effect of time correlation is seen to add a negative correction to the scaling exponents highlighting an increase of intermittency.

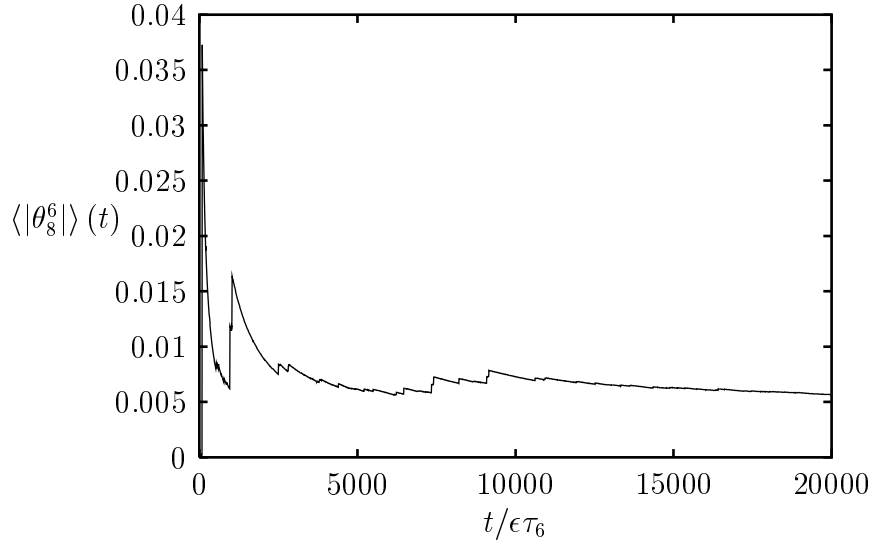


FIG. 6. The convergence of the sixth order structure function for $\epsilon = 1$. Shown is $\langle |\theta_m^6| \rangle(t)$ for $m = 8$. The fast upwards changes and slow downward relaxations reveal the intermittent nature of the signal $\theta_{m=8}^6(t)$

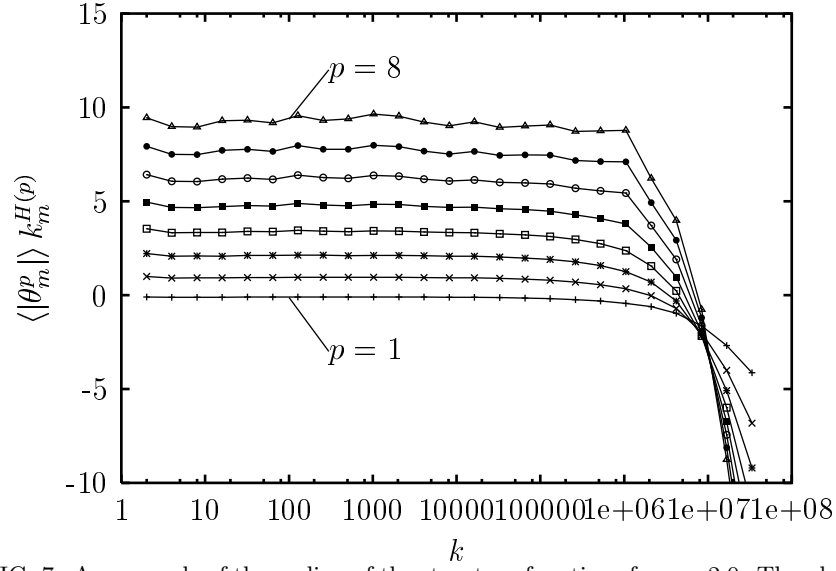


FIG. 7. An example of the scaling of the structure functions for $\epsilon = 2.0$. The plot shows the structure functions “normalised” by the fitted scaling $k_m^{H(p)}$ to make the scaling regime appear as horizontal lines. The lower line is for $p = 1$, and the upper line is $p = 8$. Each line is offset to make it possible to distinguish the lines from each other.

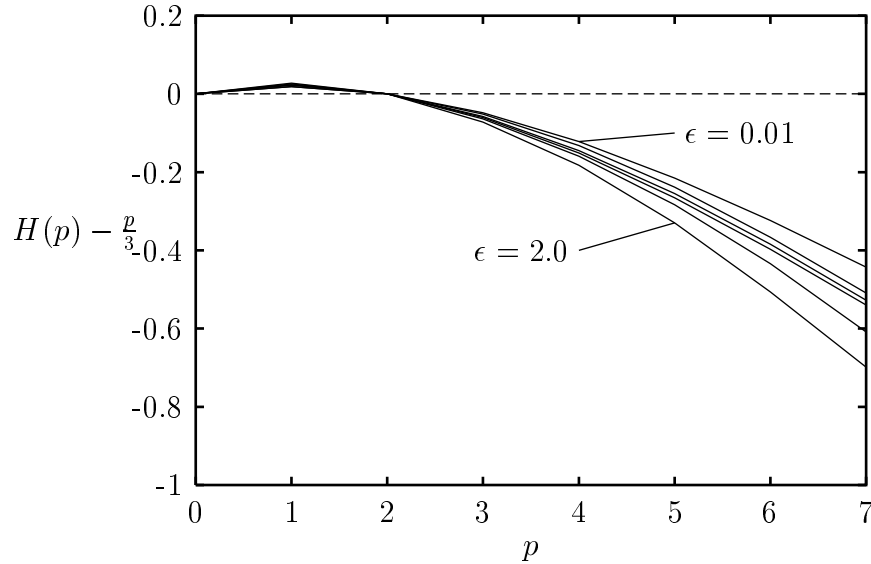


FIG. 8. The anomalous part of the structure functions $H(p) - p/3$ as a function of p for $\epsilon = 0.01$ to $\epsilon = 2$. Also shown are the points from the analytical calculation of the scaling for the white noise case. The lines correspond to (from the top): $\epsilon = 0.01, 0.02, 0.10, 0.25, 1.0$ and 2.0 . The dashed line corresponds to normal scaling.

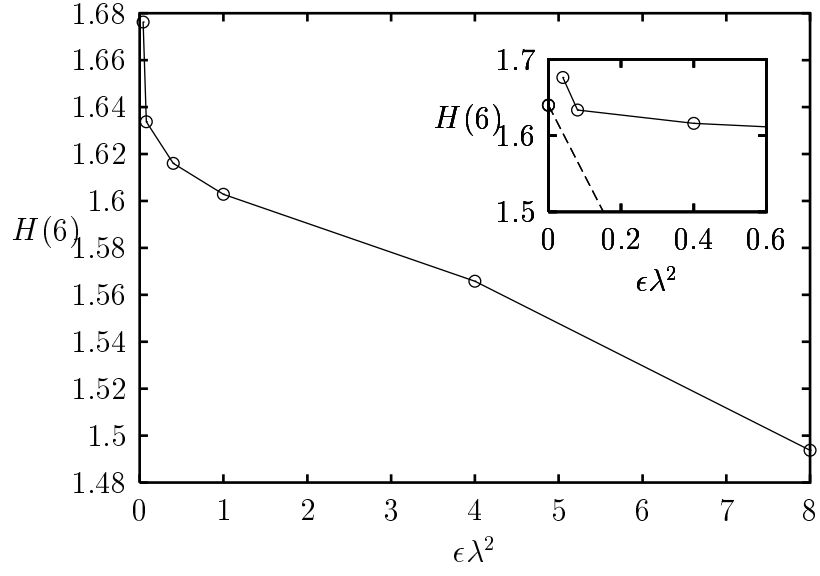


FIG. 9. The scaling of the 6.th order structure function vs. $\epsilon \lambda^2$. The inset shows an enlargement of the perturbative range $\epsilon \lambda^2 \ll 1$ where the analytical prediction from first order perturbation theory can be compared with the numerical experiments.